THE FIXED POINT INDEX AND SOME APPLICATIONS

ROGER D. NUSSBAUM

SMS

SÉMINAIRE DE MATHÉMATIQUES SUPÉRIEURES
SÉMINAIRE SCIENTIFIQUE OTAN (NATO ADVANCED STUDY INSTITUTE)
DÉPARTEMENT DE MATHÉMATIQUES ET DE STATISTIQUE — UNIVERSITÉ DE MONTRÉAL

THE FIXED POINT INDEX AND SOME APPLICATIONS

ROGER D. NUSSBAUM
Rutgers University

1985
LES PRESSES DE L'UNIVERSITÉ DE MONTRÉAL
C.P. 6128, succ. «A», Montréal (Québec), Canada H3C 3J7

ISBN 2-7606-0711-9
DÉPÔT LÉGAL — 2e TRIMESTRE 1985 — BIBLIOTHÈQUE NATIONALE DU QUÉBEC

CONTENTS

INTRODUCTION

If X is an appropriate topological space, W an open subset of X and $f : W \to X$ a suitable map, one can define an integer, $i_X(f,W)$, called the fixed point index of $f : W \to X$. The integer $i_X(f,W)$ is, roughly speaking, an algebraic count of the number of fixed points of f in W. If X is a Banach space, W is bounded, and $f : \bar{W} \to X$ is a compact map such that $x \neq f(x)$ for $x \in \partial W$, $i_X(f,W)$ agrees with the Leray-Schauder degree of $I - f$ on W with respect to 0. If $W = X =$ a compact polyhedron, $i_X(f,X)$ agrees with the Lefschetz number of $f : X \to X$, and this agreement provides a chance to use powerful results from algebraic topology.

The purpose of these expository notes is to provide an introduction to the classical fixed point index of algebraic topology [2,16], describe some generalizations which seem appropriate for applications in analysis and illustrate the usefulness of the fixed point index in analysis with some examples. Although the Leray-Schauder degree is widely known and appreciated among analysts, the level of awareness about the fixed point index, which is a natural generalization of Leray-Schauder degree, is much lower. I hope these notes will serve as propaganda for the usefulness of the fixed point index and fixed point theorems in analysis.

Of course, as many practising analysts well know, neither the Leray-Schauder degree nor the fixed point index is a panacea. To use them to obtain existence results it may be necessary to obtain difficult *a priori*-bounds as, for example, in [34]. If one obtains existence of solutions, it may still be a difficult problem to prove qualitative properties of solutions (as is, indeed, the case for the example discussed in Section 5). Nevertheless, the fixed point index provides a useful starting point.

This paper is rather long, so an outline may be in order. The first section starts from the topological degree in $\mathbf{R}^n$ and provides a summary of the Leray-Schauder degree and the fixed point index. An attempt is made to give some indication of proofs, but generally speaking, complete proofs are not given. However, in perhaps the most important example for analysis, namely when X is a closed convex subset of a Banach space Y, W is a relatively open subset of X and $f : W \to X$ is locally compact, it is shown how $i_X(f,W)$ can be directly defined in terms of Leray-Schauder degree.

Section 2 provides an application of these ideas to linear functional analysis. It is shown how the most general version of the famous linear Krein-Rutman theorem [76] can be obtained with the fixed point index. The fixed point approach to this circle of theorems has a long history [13, 103, 111], including some attempts in the original Krein-Rutman paper; but it seems that the Schauder fixed point theorem and Leray-Schauder degree theory are insufficient to obtain the most general versions of the Krein-Rutman theorem.

The third section of the paper returns to fixed point theory. Following a suggestion of A.J. Tromba, a proof is given of the so called "mod p theorem" of Steinlein, Krasnosel'skii and Zabreiko. This result is then used to compute the fixed point index of a map f on a small neighborhood of a so-called "ejective fixed point" of f. Analogous results are obtained for "attractive fixed points". These theorems are all examples of "asymptotic fixed point theorems", in which information about fixed points of a map f is obtained from information about iterates f^p of f.

The fourth section considers a locally compact map $f : J \times X \to X$, where J is an open interval of reals, X is a "metric ANR" (e.g., a closed, convex subset of a Banach space) and f is a locally compact map such that $f(\lambda,x_o) = x_o$ for all $\lambda \in J$ and some x_o. In this framework, various generalizations of Rabinowitz's famous global bifurcation theorem are proved. In one corollary, it is assumed that x_o is an attractive fixed point of f_λ $(f_\lambda(x)=f(\lambda,x))$ for $\lambda < \lambda_o$ and an ejective fixed point for $\lambda > \lambda_o$, and the results of Section 3 are used. It should be noted that f_λ is not usually assumed differentiable in any sense at x_o, and, in the particular example given in Section 5, is *not* differentiable at x_o.

In Section 5 the problem of proving unbounded, connected sets $\{(\lambda,x)\}$ of periodic solutions of

$$\dot{x}(t) = -\lambda g(x(t)) - \lambda f(x(t-1))$$

is studied with the aid of Corollary 4.1. The main result is Theorem 5.1 (and the corollaries after it), which generalizes an existence result for periodic solutions obtained by Mallet-Paret and the author in [86]. However, the real interest in [86] was in studying qualitative behavior of periodic solutions as $\lambda \to \infty$. The theorems of Section 4 provide only a starting point by enabling one to prove existence of solutions.

Section 1

THE TOPOLOGICAL DEGREE AND THE FIXED POINT INDEX OF MAPPINGS

I would like to begin by recalling the basic facts about the topological degree of a mapping and by indicating the relation of topological degree to the fixed point index and Lefschetz fixed point theorem of algebraic topology. Almost immediately one semantic point must be made: some analysts use the term "index" to refer to the topological degree of a mapping F at an isolated solution of an equation $F(x) = a$. This is emphatically not the sense in which the term "index" will be used here; as will be seen later, the fixed point index is a generalization of the Leray-Schauder topological degree.

For the most part I shall give no proofs in this introductory section. My reasons are twofold. First, details can be found in a variety of sources, e.g., [2], [16], [30], [40], [56], [89], [90], [117]. Second, the topological degree and the fixed point index satisfy certain properties which determine them axiomatically. Thus, in some sense, one knows everything about the topological degree and the fixed point index if one knows the properties which determine them axiomatically. Needless to say, the previous statement must be taken with a grain of salt.

To describe the topological degree in the simplest case, recall that if U is a bounded, open subset of $\mathbf{R}^n$ and $F : \overline{U} \to \mathbb{R}^n$ is a continuous map such that, for some $a \in \mathbf{R}^n$,

$$a \notin F(\partial U) ,$$

then one can define the topological degree of F on U with respect to a, written $\deg(F,U,a)$. The topological degree is an integer and can be considered intuitively as an algebraic count of the number of solutions $x \in U$ of the equation $F(x) = a$. More precisely, the degree can be defined as follows: Select $\delta > 0$ such that $\|F(x)-a\| \geq \delta$ for all $x \in \partial U$. Let $G : \overline{U} \to \mathbf{R}^n$ be a continuous function such that G is C^1 on U and $\|F(x)-G(x)\| < \delta$ for all $x \in \partial U$. According to Sard's theorem, G has a regular value $b \in \mathbf{R}^n$ arbitrarily close to a; in particular, b can be chosen so that

(1.1) $$\sup_{x \in \partial U} \|F(x)-G(x)\| < \delta - \|b-a\| .$$

(Recall that a regular value b of G is such that the Jacobian matrix $J_G(x)$, is nonsingular for all $x \in G^{-1}(b) \cap U$.) The inequality (1.1) implies that the equation $G(x) = b$ has no solutions $x \in \partial U$, so $\{x \in \overline{U} | G(x)=b\} = S$ is a compact subset of U. Because b is a regular value, the implicit function theorem implies that each element of S is isolated, so S must be a finite set, say,

$$S = G^{-1}(b) = \{x_1, x_2, \ldots, x_p\} .$$

If $\operatorname{sgn}(\det J_G(x))$ denotes the sign (± 1) of the determinant of $J_G(x)$, one *defines* $\deg(F,U,a)$ by

(1.2) $$\deg(F,U,a) = \sum_{x \in G^{-1}(b)} \operatorname{sgn}(\det J_G(x)) .$$

The difficulty, of course, is to prove that the previous definition is independent of the choice of G and b as above, and I shall assume this has been done.

It is easy to prove that if $U_1 \subset U$ is any open neighborhood of $\{x \in U | F(x)=a\} = \Sigma$, then

$$\text{(1.3)} \qquad \deg(F,U_1,a) = \deg(F,U,a)$$

(Recall that we assume $F(x) \neq a$ for $x \in \partial U$, so Σ is compact). Because of this one can give a slight, but convenient, generalization of degree. If W is an open subset of $\mathbf{R}^n$ (possibly unbounded) and $F : W \to \mathbf{R}^n$ is a continuous map such that $\Sigma = \{x \in W | F(x)=a\}$ is compact (possibly empty), then one can *define*

$$\text{(1.4)} \qquad \deg(F,W,a) = \deg(F,W_1,a) ,$$

where W_1 is any bounded, open neighborhood of Σ such that $\bar{W}_1 \subset W$. The right hand side of equation (1.4) has already been defined, and if one uses equation (1.3) one can see that the definition is independent of the particular W_1 chosen.

It is also convenient to make the convention that $\deg(F,W,a)$ is defined and equal to zero whenever W is the empty set.

With the above definitions one can easily verify that the topological degree satisfies the following three properties:

1. If W is an open subset of $\mathbf{R}^n$, $I : \mathbf{R}^n \to \mathbf{R}^n$ denotes the identity map, and $a \in \mathbf{R}^n$, then $\deg(I,W,a) = 1$ if $a \in W$ and $\deg(I,W,a) = 0$ if $a \notin W$.

2. Suppose that W is an open subset of $\mathbf{R}^n$, $F : W \to \mathbf{R}^n$ is a continuous map, $a \in \mathbf{R}^n$ and $S = \{x \in W | F(x)=a\}$ is compact (possibly empty). Let W_1 and W_2 be disjoint open subsets of W (W_1 or W_2 may be empty) such that $S \subset W_1 \cup W_2$. Then one has

$$\deg(F,W,a) = \deg(F,W_1,a) + \deg(F,W_2,a) .$$

3. Suppose that W is an open subset of $\mathbf{R}^n$, $a \in \mathbf{R}^n$, and $F : W \times [0,1] \to \mathbf{R}^n$ is a continuous map such that $\{x \in W | F(x,t)=a \text{ for some } t \in [0,1]\}$ is compact. If $F_t(x) := F(x,t)$, then $\deg(F_t,W,a)$ is defined and constant for $0 \le t \le 1$.

The first of the preceding three properties of the topological degree is customarily called the normalization property, the second property is called the additivity property and the third the homotopy property.

Conversely, suppose that, for each ordered triple (F,W,a) such that $a \in \mathbf{R}^n$, W is an open subset of $\mathbf{R}^n$ (possibly empty), and $F : W \to \mathbf{R}^n$ is a continuous map for which $\{w \in W | F(x)=a\}$ is compact (possibly empty), there exists an integer-valued function $\deg(F,W,a)$ which satisfies the normalization, additivity and homotopy properties. L. Führer [48] and H. Amann and S. Weiss [7] have proved that such a function $\deg(F,W,a)$ is unicue. The key step is to prove that if $\theta : \mathbf{R}^n \to \mathbf{R}^n$ is affine linear, W is an open subset of $\mathbf{R}^n$ and $a \in \theta(W)$, then

$$\deg(\theta,W,a) = \operatorname{sgn}(\det(\theta)) ,$$

where $\det(\theta)$ is the determinant of θ and $\operatorname{sgn}(u) = +1$ if $u > 0$ and $\operatorname{sgn}(u) = -1$ if $u < 0$.

In the 1930's, Leray and Schauder [80] observed that the above topological degree could be extended to a Banach space setting, but at the cost of restricting the class of allowable functions F. Recall that if D is a topological space and $f : D \to Z$ is a continuous map into a topological space Z, then f is called compact if the closure of $f(D)$ in Z is compact. If, for each $x \in D$, there exists an open neighborhood N_x of x such that $f|N_x$ is compact, then f will be called locally compact. If D is a subset of a normed linear space X and $f : D \to X$ is a compact map, Leray and Schauder proved [80] that there exists a sequence of continuous, compact maps $f_n : D \to X_n$, where X_n is a finite dimensional subspace of X and $\lim_{n\to\infty}(\sup_{x\in D}\|f(x)-f_n(x)\|) = 0$. The proof is a simple partition of unity argument.

Now suppose that W is an open subset of a normed linear space X and that $f : W \to X$ is a continuous map such that $\{x\in W : x-f(x)=0\} = S$ is compact (possibly empty). Assume that there exists an open neighborhood V of S such that $\overline{V} \subset W$ and $f|\overline{V}$ is compact. (Note that V will exist if f is locally compact.) The compactness of f and the fact that $x \neq f(x)$ for $x \in \partial V$ imply that there exists $\delta > 0$ such that

$$\inf\{\|x-f(x)\| : x\in\partial V\} = \delta > 0 .$$

The preceding remarks imply that there exists a continuous, compact map $g : \overline{V} \to X$ such that $g(V) \subset Y$, Y a finite dimensional subspace of X, and

$$\sup_{x\in V}\|f(x)-g(x)\| < \delta .$$

If I is the identity map, $F = I - f$ and $G = (I-g)|(V\cap Y)$, one defines the Leray-Schauder degree, $\deg(F,W,0)$, by

$$\deg(F,W,0) = \deg(G,V\cap Y,0) . \tag{1.5}$$

It is not hard to show that this definition is independent of the particular open set V and mapping g . Of course the degree on the right hand side of equation (1.5) has already been defined, because Y is finite dimensional.

The above definition gives the degree of F on W with respect to 0 . If $a \in X$, W is an open subset of X and $f : W \to X$ is a continuous map such that $\{x \in W \mid x - f(x) = a\} = S$ is compact and such that $f|\overline{V}$ is compact for some open neighborhood V of S with $\overline{V} \subset S$, one defines $\deg(I-f,W,a)$ by

$$\deg(I-f,W,a) = \deg(I-f-a,W,0) \quad (1.6)$$

In equation (1.6), $I - f - a$ denotes the map $x \to x - f(x) - a$.

The properties of the Leray-Schauder degree follow directly from the corresponding properties for finite dimensional degree. The normalization property is exactly as before and will not be repeated. The additivity property takes the following form:

Suppose that W is an open subset of a normed linear space X , $a \in X$ and $f : W \to X$ is a continuous map such that $S = \{x \in W \mid x - f(x) = a\}$ is compact (possibly empty) and such that there exists an open neighborhood V of S with $F|V$ compact. If W_1 and W_2 are disjoint open subsets of W (W_1 or W_2 possibly empty) and $S \subset W_1 \cup W_2$, one has

$$\deg(I-f,W,a) = \deg(I-f,W_1,a) + \deg(I-f,W_2,a) .$$

Recall that one allows W to be the empty set, in which case the additivity property implies that $\deg(I-f,W,a) = 0$. In general, if $\deg(I-f,W,a)$ is nonzero, the additivity property implies that the equation $x - f(x) = a$ has a solution $x \in W$; the converse, of course, is false.

To describe the homotopy property for Leray-Schauder degree, suppose that Ω is an open subset of $X \times [0,1]$, where X is a normed linear space, and that $f : \Omega \to X$ is a continuous map. Assume that $\Sigma = \{(x,t) \in \Omega \mid x - f(x,t) = a\}$ is compact (possibly empty) and that there exists an open neighborhood V of Σ (open as a subset of $X \times [0,1]$) such that $F|V$ is compact. If $f_t(x) = f(x,t)$ and $\Omega_t = \{x \in X : (x,t) \in \Omega\}$, one has $\deg(I-f_t, \Omega_t, a)$ is constant for $0 \le t \le 1$.

The Leray-Schauder degree is frequently described in a somewhat less general framework. One assumes that W is a bounded open subset of a Banach space X and that $f : \overline{W} \to X$ is a compact map such that $x - f(x) \neq a$ for all $x \in \partial W$. It then follows that $S = \{x \in W : x - f(x) = a\}$ is compact and that f is compact on an open neighborhood of S (namely W), so the previous definition of Leray-Schauder degree applies.

The homotopy property for Leray-Schauder degree also takes a much simpler looking form if one strengthens hypotheses. Assume that W is a bounded, open subset of a normed linear space X and that $f : \overline{W} \times [0,1] \to X$ is a compact map such that $x - f(x,t) \neq a$ for $(x,t) \in \partial W \times [0,1]$. One can then prove that $S = \{(x,t) \in W \times [0,1] : x - f(x,t) = a \text{ for some } t \in [0,1]\}$ is compact, so the hypotheses of the general homotopy property imply that $\deg(I-f_t, W, a)$ is constant for $0 \le t \le 1$, where $f_t(x) = f(x,t)$.

One can easily show, by using the corresponding uniqueness result for the topological degree in $\mathbf{R}^n$, that the normalization, additivity and homotopy properties uniquely determine the Leray-Schauder degree: see [7].

There have been many generalizations of Leray-Schauder degree, both with regard to the class of functions F allowed and to the class of spaces X allowed. In these notes I shall be interested in taking $a = 0$, so $\deg(I-f,W,a)$

can be considered as an algebraic count of the number of fixed points of f in W, where W is an open subset of a normed linear space X. I want to generalize this situation initially by considering a Hausdorff topological space X, an open subset W of X, and a continuous map $f : W \to X$ such that $S = \{x \in W | f(x) = x\}$ is compact and such that there exists an open neighborhood V of S for which $f|V$ is compact. In this situation and for a suitable class of spaces X, I want to define an integer, which I will denote $i_X(f,W)$ and which will be called the fixed point index of f on W. The fixed point index will have properties analogous to those of the topological degree. In fact, if X is a normed linear space, $i_X(f,W)$ is defined by

$$i_X(f,W) = \deg(I-f,W,0) .$$

The first question is what class of spaces X to consider, and a natural class from our viewpoint is the set of metrizable absolute neighborhood retracts or ANR's. A metric space X is called an ANR if, whenever it is homeomorphic to a closed subset X_1 of a metric space M, there exists an open neighborhood U of X_1 in M and a continuous retraction r of U onto X_1 (so $r(U) \subset X_1$ and $r(y)=y$ for all $y \in X_1$). A metrizable space X is called an absolute retract or AR if whenever X is homeomorphic to a closed subset X_1 of a metric space M, there exists a continuous retraction r of M onto X_1. O. Hanner [60,61] has proved that if X is a metric ANR, A is a closed subset of a metric space M and $g : A \to X$ is a continuous map, then there exists an open neighborhood W of A in M and a continuous extension $G : W \to X$ of g (so $G(a)=g(a)$ for all $a \in A$). If X is an AR, there exists a continuous extension $G : M \to X$ of g. Conversely, metric spaces with these extension properties are obviously ANR's or AR's.

A result of J. Dugundji [39] implies that if C is a convex subset of a normed linear space Y, then C is an AR. If C is also a closed subset of Y, there exists a retraction r of Y onto C. Dugundji's theorem and some general results about ANR's [14] imply that if $C_1, C_2, \ldots, C_n$ are closed, convex subsets of a normed linear space Y and $X = \bigcup_{j=1}^{n} C_j$, then X is an ANR. More generally, if X is a closed subset of a normed linear space Y, and if there exists a family $\{C_j : j \in J\}$ of closed, convex subsets of Y such that $\{C_j : j \in J\}$ is a locally finite covering of X, then X is an ANR.

It will be useful later to have some notation describing the previous two examples.

DEFINITION 1.1. If X is a closed subset of some Banach space Y (from which X inherits its metric) and if there exist finitely many closed, convex subsets $C_1, C_2, \ldots, C_n$ of Y such that $X = \bigcup_{j=1}^{n} C_j$, then we shall write $X \in F_0$. If X is a closed subset of some Banach space Y and if there exists a family $\{C_j : j \in J\}$ of closed, convex subsets C_j of Y such that $X = \bigcup_{j \in J} C_j$ and such that $\{C_j : j \in J\}$ is a locally finite covering of X, then we shall write $X \in F$.

There are many examples of metric ANR's other than unions of convex sets. If X is a metric ANR, $X_1 \subset X$ and there exists a continuous retraction of X onto X_1, then X_1 is an ANR; more succinctly, a retract of an ANR is an ANR. The proof is trivial. Similarly, one can easily show that a homeomorphic image of an ANR is an ANR, and any open subset of an ANR is an ANR. The latter fact implies that any ANR is "locally" an ANR. O. Hanner [60,61] has proved a beautiful and deep converse result: if X is a metric space and every point $x \in X$ is contained in an open neighborhood N_x which is an ANR, then X is an ANR. Note (see [61]) that X need not be separable. In particular, a metrizable Banach manifold is an ANR.

Now suppose that X is a metric ANR, W is an open subset of X and $f : W \to X$ is a continuous map such that $S = \{x \in W : f(x)=x\}$ is compact (possibly empty) and such that there exists an open neighborhood V of S for which $f|V$ is compact. A result of Arens and Eells [8] asserts that there exists an embedding j_1 of X as a closed subset $X_1 = j_1(X)$ of a normed linear space Y_1. Because X is an ANR, there exists an open neighborhood O_1 of X_1 and a continuous retraction r_1 of O_1 onto X_1. If one defines $W_1 = j_1(W)$, $V_1 = j_1(V)$, $S_1 = j_1(S)$ and $f_1 = j_1 f j_1^{-1} : W_1 \to X_1$, one can easily check that S_1 is the fixed point set of f_1 in W_1, and that V_1 is an open neighborhood of S_1 in X_1 and $f_1|V_1$ is compact. It is not hard to see that the map $f_1 r_1 : r_1^{-1}(W_1) \to X_1 \subset Y_1$ also has S_1 as its fixed point set and is compact on $r_1^{-1}(V_1)$.. Thus one can consider

$$\deg(I-f_1 r_1, r_1^{-1}(W_1), 0) .$$

It seems to have been J. Leray who first observed (at least for compact metric ANR's) that the preceding construction provides a means of defining the fixed point index, namely, one *defines* $i_X(f,W)$, the fixed point index of f on W, by

$$i_X(f,W) = \deg(I-F_1 r_1, r_1^{-1}(W_1), 0) . \tag{1.7}$$

Of course the difficulty in this approach is to prove that the left hand side of eq. (1.7) is independent of the embedding j_1 and the retraction r_1. In order to prove this, one needs a property of the Leray-Schauder degree which is important but not well-known.

THEOREM. (The commutativity property). *Suppose that* Y_1 *and* Y_2 *are normed linear spaces, that* U_j , $j = 1,2,$, *is an open subset of* Y_j *and that* $g_1 : U_1 \to Y_2$ *and* $g_2 : U_2 \to Y_1$ *are continuous maps. Consider the maps*

$$g_2 g_1 : g_1^{-1}(U_2) \subset Y_1 \to Y_1$$

and

$$g_1 g_2 : g_2^{-1}(U_1) \subset Y_2 \to Y_2$$

and their corresponding fixed point sets

$$S = \{y_1 \in g_1^{-1}(U_2) : g_2 g_1(y_1) = y_1\}$$

and

$$T = \{y_2 \in g_2^{-1}(U_1) : g_1 g_2(y_2) = y_2\} .$$

Assume that S *or* T *is compact (possibly empty) and that* g_1 *is compact on some open neighborhood of* S *or* g_2 *is compact on some open neighborhood of* T . *Then one has*

$$\deg(I - g_2 g_1, g_1^{-1}(U_2), 0) = \deg(I - g_1 g_2, g_2^{-1}(U_1), 0) .$$

In particular $S = g_2(T)$ *and* $T = g_1(S)$ *are compact and both Leray-Schauder degrees are defined.*

The proof of the commutativity property is not hard, but it is somewhat long and tedious and will be omitted here. A basic step in the proof is the following simple linear algebra lemma.

LEMMA. *If* A *and* B *are linear maps of* $\mathbb{R}^n$ *into* $\mathbb{R}^n$, *then*

$$\det(I - BA) = \det(I - AB) .$$

PROOF. If B is invertible, one obtains

$$\det(I-BA) = \det(B(B^{-1}-A))$$
$$= \det((B^{-1}-A)B) = \det(I-AB) .$$

If B is not invertible, select a sequence $\varepsilon_n \to 0$ such that $\varepsilon_n I + B$ is invertible. The previous argument gives

$$\det(I-(B+\varepsilon_n I)A) = \det(I-A(B+\varepsilon_n I)) ,$$

and the lemma follows by taking limits as $\varepsilon_n \to 0$.

Once one has proved the commutativity property for Leray-Schauder degree, one can show that the right hand side of eq. (1.7) is independent of the embedding j_1 and the retraction r_1 . To see this, let X , W and f be as in the sentences immediately preceding eq. (1.7) and let j_k $(k=1,2)$ be embeddings of X as closed subsets $j_k(X)$ of normed linear spaces Y_k $(k=1,2)$. Let r_k $(k=1,2)$ be continuous retractions of open neighborhoods O_k of $j_k(X)$ in Y_k onto Y_k . If one writes $X_k = j_k(X)$ and $W_k = j_k(W)$, the problem is to prove that

$$\deg(I-j_1fj_1^{-1}r_1, r_1^{-1}(W_1), 0) = \deg(I-j_2fj_2^{-1}r_2, r_2^{-1}(W_2), 0) . \tag{1.8}$$

To prove eq. (1.8), define mappings g_1 and g_2 by

$$g_1 = j_2fj_1^{-1}r_1 : r_1^{-1}(W_1) \to Y_2 \quad \text{and}$$
$$g_2 = j_1j_2^{-1}r_2 : O_2 \to Y_1 .$$

One can easily prove that

$$j_2fj_2^{-1}r_2 = g_1g_2$$

on $r_2^{-1}(W_2)$ and that

$$j_1 f j_1^{-1} r_1 = g_2 g_1$$

on $r_1^{-1}(W_1)$, and equation (1.8) then follows from the commutativity property.

There is one important case in which the preceding discussion can be simplified enormously (as was observed in [97]). Suppose that X is a closed, convex subset of a normed linear space Y , that W is a relatively open subset of X and that $f : W \to X$ is a continuous map such that $S = \{x \in W : f(x) = x\}$ is compact (possibly empty) and there exists an open neighborhood V of S , $V \subset W$, such that $f|V$ is compact. Dugundji's theorem [39] implies that there exists a continuous retraction $r_1 : Y \to X$, and as already discussed one can define

$$i_X(f,W) = \deg(I - fr_1, r_1^{-1}(W), 0) .$$

However, in this case it is easy to show that the definition is independent of the particular retraction. To see this, suppose that $r_2 : Y \to X$ is a continuous retraction of Y onto X ; one must prove that

$$\deg(I - fr_1, r_1^{-1}(W), 0) = \deg(I - fr_2, r_2^{-1}(W), 0) . \tag{1.9}$$

If $f(r_k x) = x$ for $x \in r_k^{-1}(W)$, then $x \in X$, so $r_k(x) = x$ and $x \in S$. Thus S is the fixed point set of fr_k in $r_k^{-1}(W)$, so if $U \subset r_1^{-1}(W) \cap r_2^{-1}(W)$ is an open neighborhood of S in Y ,

$$\deg(I - fr_k, U, 0) = \deg(I - fr_k, r_k^{-1}(W), 0) ,$$

and it suffices to prove

$$\deg(I - fr_1, U, 0) = \deg(I - fr_2, U, 0) . \tag{1.10}$$

Select U to be a sufficiently small neighborhood of S in Y so that $(1-t)r_1(x) + tr_2(x) \in W$ for all $x \in U$ and $0 \le t \le 1$; this can be done because

X is convex and because $(1-t)r_1(x) + tr_2(x) = x$ for all $x \in X$. Consider the homotopy

$$x \to f((1-t)r_1(x)+tr_2(x)) = f_t(x) \ , \ 0 \le t \le 1 \ ,$$

for $x \in U$. If one can prove that $f_t(x) = x$ (for $0 \le t \le 1$ and $x \in U$) if and only if $x \in X$, the homotopy property for Leray-Schauder will give eq. (1.10). However, if $f_t(x) = x$ for $x \in U$, then $x \in X$ (because $f(W) \subset X$), so

$$(1-t)r_1(x) + tr_2(x) = x \in W \ ,$$

and $f(x) = x$, that is, $x \in S$.

Once one has defined the fixed point index, one can verify directly that it satisfies analogues of the additivity, homotopy and commutativity properties of the Leray-Schauder degree. More precisely, one has

1. (The additivity property). Let X be a metric ANR, W an open subset of X and $f : W \to X$ a continuous map such that $S = \{x \in W : f(x) = x\}$ is compact (possibly empty) and f is compact on some open neighborhood V of S. Let W_1 and W_2 be disjoint open subsets of W (possibly empty) such that $S \subset W_1 \cup W_2$. Then one has

$$i_X(f,W) = i_X(f,W_1) + i_X(f,W_2) \ . \tag{1.11}$$

As usual, if S is empty, the additivity property implies that $i_X(f,W) = 0$, so that if one ever has $i_X(f,W) \neq 0$, S must be nonempty. Also, by taking W_2 to be the empty set in eq. (1.11) and W_1 to be any open neighborhood of S with $W_1 \subset W$, one obtains

$$i_X(f,W) = i_X(f,W_1) .$$

The latter fact is frequently useful.

2. (The homotopy property). Let X be a metric ANR, Ω an open subset of $X \times [0,1]$, and $f : \Omega \to X$ a continuous map such that $\Sigma = \{(x,t) \in \Omega : f(x,t) = x\}$ is compact and f is compact on some open neighborhood V of Σ in Ω. If $f_t(x) = f(x,t)$ and $\Omega_t = \{x \mid (x,t) \in \Omega\}$, then $i_X(f_t, \Omega_t)$ is constant for $0 \le t \le 1$.

The standard formulation of the homotopy property is to take $\Omega = W \times [0,1]$, where W is an open subset of X, and to assume that $f : \overline{W} \times [0,1] \to X$ is a compact map such that $f(x,t) \neq x$ for $(x,t) \in \partial W \times [0,1]$. One then concludes that $i_X(f_t, W)$ is constant for $0 \le t \le 1$. Of course this is a special case of the general version of the homotopy property, but conversely, the general version can be derived from this special case.

3. (The commutativity property). Let W_k be an open subset of a metric ANR X_k, $k = 1,2$, and suppose that $f_1 : W_1 \to X_2$ and $f_2 : W_2 \to X_1$ are continuous maps. Let $S = \{x \in f_1^{-1}(W_2) : f_2 f_1(x) = x\}$ and $T = \{y \in f_2^{-1}(W_1) : f_1 f_2(y) = y\}$ and assume that S is compact (so $T = f_1(S)$ is also) and that f_1 is compact on some open neighborhood of S, or f_2 is compact on an open neighborhood of T. Then one has

$$i_{X_1}(f_2 f_1, f_1^{-1}(W_2)) = i_{X_2}(f_1 f_2, f_2^{-1}(W_1)) .$$

The most frequent application of the commutativity property will be to the following situation: Suppose that X is a metric ANR, W is an open subset of X and $f : W \to X$ is a continuous map such that $i_X(f,W)$ is defined. If $Y \subset X$ is a metric ANR such that the inclusion $j : Y \to X$ is continuous and if

$f(W) \subset Y$, then

$$i_X(f,W) = i_Y(f,W \cap Y) \quad (1.12)$$

In eq. (1.12), notation has been abused by using the letter f to refer to the map $f : W \to X$ and also the map $f : W \cap Y \to Y$, but the meaning is clear. To prove eq. (1.12), write $f_2 = j$ and write $f_1 : W \to Y$ for the map given by $f_1(x) = f(x)$, $x \in W$, i.e., f considered as a map from W to Y . The commutativity property implies that

$$i_X(f_2 f_1, W) = i_Y(f_1 f_2, W \cap Y) ,$$

which gives eq. (1.12).

The analogue for the fixed point index of the normalization property of the topological degree is much less obvious and is essentially the Lefschetz fixed point theorem. From the viewpoint adopted here, the Lefschetz fixed point theorem is the assertion that the Lefschetz number, $\Lambda(f)$, of a map f , which is defined in terms of homology groups, agrees with the fixed point index of f when both are defined. Thus, if $\Lambda(f) \neq 0$, the map f will have a fixed point. One can prove the equality of Lefschetz number and fixed point index in the case of a compact polyhedron by using H. Hopf's device for proceeding from the homology level to the level of simplicial chains (see [16], Chapter 1) and then use some geometrical arguments to obtain the case of metric ANR's. A complete proof would be too long to give here.

The following is a precise statement of the normalization property.

4. (The normalization property). Let X be a compact metric ANR and $f : X \to X$ a continuous map. Then $H_i(X)$ (singular homology with rational coefficients) is a finite dimensional vector space for all i and has dimension zero

for i sufficiently large. Thus it makes sense to consider $tr(f_{*i})$, the trace of the linear endomorphism $f_{*i} : H_i(X) \to H_i(X)$, and to define $\Lambda(f)$, the Lefschetz number of f, by

$$\Lambda(f) = \sum_{i \geq 0} (-1)^i tr(f_{*i}) .$$

With this notation one has

$$\Lambda(f) = i_X(f,X) ,$$

so if $\Lambda(f) \neq 0$, f has fixed point in X.

There have been many generalizations of the fixed point index and of the Lefschetz fixed point theorem; a sampling of results can be found in [23], [42], [56], [82] and [95]. For the most part, the fixed point index for locally compact maps defined on open subsets of metric ANR's will be adequate for the applications in these notes; in fact, the metric ANR will usually be a closed, convex subset of a Banach space.

For one application, however, it will be convenient to have a fixed point index for a more general class of maps. For this purpose, suppose that Y is a Banach space and that β is a map which assigns to each bounded subset A of Y a nonnegative real number $\beta(A)$. For bounded sets A and B in Y, let $\overline{co}(A)$ denote the smallest closed convex set which contains A (the "convex closure of A") and let $A + B = \{a+b : a \in A, b \in B\}$. The map β will be called "a generalized measure of noncompactness" if β satisfies the following properties:

1) $\beta(A) = 0$ if and only if the closure of A is compact.

2) For all bounded sets $A \subset Y$, $\beta(\overline{co}(A)) = \beta(A)$.

3) For all bounded sets A and B in Y, $\beta(A+B) \leq \beta(A) + \beta(B)$,

4) For all bounded sets A and B in Y, $\beta(A \cup B) = \max(\beta(A), \beta(B))$.

The idea of a measure of noncompactness was apparently first introduced by Kuratowski [77] in the context of a complete metric space Z with metric ρ. If A is a bounded subset of Z, Kuratowski defined $\alpha(A)$, the measure of noncompactness of A, by

$$\alpha(A) = \inf\{d>0 : \text{there exist finitely many sets } A_1, A_2, \ldots, A_m \text{ such that } A = \bigcup_{j=1}^{m} A_j \text{ and } \text{diameter}(A_j) \le d \text{ for } 1 \le j \le m\}. \tag{1.13}$$

Using this concept, Kuratowski proved that if $\{B_j : j \ge 1\}$ is a decreasing sequence of closed, nonempty sets such that $\lim_{j\to\infty} \alpha(B_j) = 0$, then $B_\infty = \bigcap_{j=1}^{\infty} B_j$ is compact, and nonempty and for any neighborhood U of B_∞, $B_j \subset U$ for all sufficiently large j. G. Darbo [33] observed that if Y is a Banach space and the metric is the norm on Y, then α satisfies the previously mentioned properties and is a generalized measure of noncompactness.

There are many other examples of generalized measures of noncompactness. For example, if (M,d) is a compact metric space, let $C(M,\mathbb{R}^n)$ denote the space of continuous maps $u : M \to \mathbb{R}^n$ with norm

$$\|u\| = \sup\{|u(t)| : t \in M\}.$$

(Here $|v|$ denotes a given norm on $\mathbb{R}^n$.) For a bounded set A define

$$\beta_\delta(A) = \sup\{|f(s)-f(t)| : f \in A, d(s,t) \le \delta\}$$

and define $\beta(A) = \lim_{\delta \to 0^+} \beta_\delta(A)$. One can prove that β is a generalized measure of noncompactness.

If f is a compact map, then f decreases the measure of noncompactness of sets in the sense that for a bounded set A in the domain of f, $\beta(f(A)) = 0 < \beta(A)$ unless $\bar{A}$ is compact. G. Darbo [33] observed that the idea of a map which decreases the measure of noncompactness of sets can be formalized and exploited quite usefully. Specifically, suppose that D is a subset of a Banach space Y_1, β_1 is a generalized measure of noncompactness on Y_1, β_2 is a generalized measure of noncompactness on a Banach space Y_2 and $f : D \to Y_2$ is a continuous map. If c is a nonnegative real number, then f will be called a "c-set-contraction" (with respect to β_1 and β_2) if for every bounded set $A \subset D$, $f(A)$ is bounded and

$$\beta_2(f(A)) \leq c\beta_1(A) \ . \tag{1.14}$$

If $Y_1 = Y_2$, it will always be assumed that $\beta_1 = \beta_2$. If the constant c in eq. (1.14) can be chosen so that $c < 1$, f will be called a "strict-set-contraction" (w.r.t. β_1 and β_2). In general, mention of β_1 and β_2 will be omitted unless confusion is likely to result. If D is a subset of a Banach space Y, $f : D \to Y$ is a continuous map and β is a generalized measure of noncompactness on Y, f will be called a "local strict-set-contraction" if for every $x \in D$ there exists a relatively open neighborhood N_x such that $f|N_x$ is a strict-set-contraction.

If f is a strict-set-contraction (local strict-set-contraction) and g is a compact map (locally compact map) then $f + g$ is a strict-set-contraction (local strict-set-contraction) and the constant c in eq. (1.14) is unchanged. Since a Lipschitz map with Lipschitz constant $c < 1$ is a c-set-contraction, one immediately generates nontrivial examples. The fact that strict-set-contractions also behave nicely with respect to partition of unity arguments and the taking of composition (see [90] for details) makes it convenient to work with such maps.

It may seem that the introduction of local strict-set-contractions is an example of generality for the sake of generality, but this is not so. In fact, in discussing the fixed point index of a map f, a more general assumption will be made: It will be assumed that f has a compact fixed point set S and that f is a strict-set-contraction on some open neighborhood V of S. If f is a local strict-set-contraction and has compact fixed point set S, then one can easily prove the existence of such a neighborhood V. The reason for this generality is that there exist many maps which are not strict-set-contractions on their entire domains but which have compact fixed point sets and are strict-set-contractions on open neighborhoods of these sets. The problem of proving a fixed point set compact is a very natural (and frequently difficult) question which in more concrete situations is sometimes called the "problem of finding *a priori* bounds". The problem of proving the existence of a neighborhood V as above is equivalent to showing that f is "nice" on a neighborhood of the fixed point set. In general (see, for example, [95]) it seems to be natural to look for a generalized fixed point index of a map f which has a compact fixed point set S and is "nice" on a neighborhood of S; the problem, of course, is to define "nice".

To illustrate how a map may fail to be a strict-set-contraction on its entire domain D, suppose that D is a subset of a Banach space Y and that if $x \in D$, then $tx \in D$ for all $t \geq 0$. Assume that $f : D \to Y$ is a continuous map, homogeneous of degree $p > 1$ (so $f(tx)=t^p f(x)$ for all $t \geq 0$, $x \in D$), and that there exists a bounded set $A \subset D$ such that $\alpha(f(A)) > 0$, where α denotes the Kuratowski measure of noncompactness. If $tA = \{tx : x \in A\}$, one obtains that $\alpha(tA) = t\alpha(A)$ and $\alpha(f(tA)) = \alpha(t^p f(A)) = t^p \alpha(f(A))$. Thus, for t large enough, one must have

$$\alpha(f(tA)) > \alpha(A) ,$$

and f is not a strict-set-contraction on D .

The situation described in the previous paragraph actually arises in simple examples. Let $X = C[0,1]$, let D denote the nonnegative functions in X , and for $\lambda > 0$ define $f : D \to D$ and $g : D \to D$ by

$$(f(u))(x) = \lambda \int_x^1 u(y)u(y-x)\,dy , \quad 0 \le x \le 1 , \text{ and}$$

$$(g(u))(x) = (f(u))(x) + 1 .$$

For any subset A of D , one has $\alpha(f(A)) = \alpha(g(A))$, but the reader can verify that if

$$A = \{1+u_n \,|\, n \ge 1 , \ u_n(x) = \sin(n\pi x)\} ,$$

then $f(A)$ is not equicontinuous, so $\alpha(f(A)) > 0$. Because f is homogeneous of degree 2, the previous remarks imply that f is not a strict-set-contraction on D . On the other hand, one can prove that g is a local strict-set-contraction and has a compact (possibly empty) fixed point set in D for each $\lambda > 0$: see [9] for further references and details.

It is also not hard to give examples of maps which have a compact fixed point set S , are strict-set-contractions on a neighborhood of S , but are not local strict-set-contractions on their domains. An artificial but simple example is provided by $Y = C[0,1]$, $D = \{u \in Y : \|u\| < 1\}$ and $(f(u))(x) = u(x)^2$.

With this preliminary motivation, I would like to conclude this section by describing how a fixed point index can be defined for local strict-set-contractions on certain metric ANR's. First, some notation: if D is a subset of a Banach space Y, $f : D \to Y$ is a continuous map and $V \subset D$, define $K_1(f,V) = K_1 = \overline{co}\, f(V)$ and $K_n(f,V) = \overline{co}\, f(V \cap K_{n-1})$ and $K_\infty(f,V) = \bigcap_{n\geq 1} K_n$. One easily verifies that $K_n \supset K_{n+1}$, $f(V \cap K_n) \subset K_{n+1}$ and $f(V \cap K_\infty) \subset K_\infty$. If $f|V$ is a c-set-contraction with respect to a generalized measure of noncompactness β and $c < 1$ and V is bounded, the properties of β imply that

$$\beta(K_n) \leq c\beta(K_{n-1}) ,$$

so that $\beta(K_n) \leq c^n \beta(V)$ and $\beta(K_\infty) = 0$.

Now suppose that Y is a Banach space, $X \subset Y$ and $X \in F$ (so X is a locally finite union of closed, convex subsets of Y; see Definition 1.1). Suppose that W is a relatively open subset of X, $f : W \to X$ is a continuous map, $S = \{x \in W : f(x) = x\}$ is compact (possibly empty) and there exists a relatively open neighborhood V of S in X (V possibly empty if S is) such that $f|V$ is a strict-set-contraction with respect to a generalized measure of noncompactness β. Let K be any compact, convex subset of Y such that $K_\infty(f,V) \subset K$ and $f(V \cap K) \subset K$. Notice that such a K exists (take $K = K_\infty(f,V)$) and that $S \subset K_\infty(f,V) \subset K$, so K is nonempty if S is. Also, if one defines $K^* = K \cap X$, then $K^* \in F_0$ (in fact K^* is a finite union of compact, convex sets), so K^* is a compact metric ANR, $V \cap K^*$ is a relatively open subset of K^*, $f : V \cap K^* \to K^*$ and S is the fixed point set of $f|V \cap K^*$. Thus the fixed point index of $f : V \cap K^* \to K^*$ is defined. One *defines* $i_X(f,W)$ by the equation

$$i_X(f,W) = i_{K^*}(f, V \cap K^*) . \tag{1.15}$$

In order to show that the previous definition is well-defined one must prove that it is independent of the particular relatively open neighborhood V of S as above and of the particular compact, convex set K as above. Proving the latter fact is crucial; without it, it is not even clear that the definition in eq. (1.15) agrees with the classical fixed point index for X a compact polyhedron. The reader is referred to [90] for the detailed proof that $i_X(f,W)$ is independent of V and K.

There is one important case in which the technical difficulties of the general situation simplify greatly. Suppose that X is a closed, convex subset of a Banach space Y, that W is a relatively open subset of X and that $f : W \to X$ is a continuous map such that $S = \{x \in W \mid f(x)=x\}$ is compact and such that there exists a relatively open neighborhood V of S in X for which $f|V$ is a strict-set-contraction. Suppose that A and B are compact, convex sets such that $K_\infty(f,V) \subset A$, $K_\infty(f,V) \subset B$, $f(V \cap A) \subset A$ and $f(V \cap B) \subset B$. By intersecting A with X and B with X, one can assume $A \subset X$ and $B \subset X$, and one wants to prove

$$i_A(f,V\cap A) = i_B(f,V\cap B) \ . \tag{1.16}$$

Once one has proved eq. (1.16), it is not hard to show that $i_X(f,W)$ is independent of V as above. To prove eq. (1.16) it suffices to prove that

$$i_A(f,V\cap A) = i_C(f,V\cap C) = i_B(f,V\cap B) \ , \tag{1.17}$$

where $C = K_\infty(f,V)$. It is enough to prove

$$i_A(f,V\cap A) = i_C(f,V\cap C) \ , \tag{1.18}$$

the proof for B being the same.

By Dugundji's theorem, let ρ be a continuous retraction of Y onto A and r a continuous retraction of Y onto C. By definition one has

$$i_A(f,V\cap A) = \deg(I-f\rho,\rho^{-1}(V\cap A),0)$$

and

$$i_C(f,V\cap C) = \deg(I-fr,r^{-1}(V\cap C),0) \ .$$

The fixed point set of $f\rho$ in $\rho^{-1}(V\cap A)$ is S, and similarly for fr. Because $\rho(x) = r(x) = x$ for $x \in S$ and because A is convex, there exists a bounded open neighborhood U of S in Y such that for $0 \le t \le 1$ and $x \in U$,

$$(1.19) \qquad (1-t)\rho(x) + tr(x) \in V \cap A .$$

The claim is that if

$$(1.20) \qquad x = f((1-t)\rho(x)+tr(x))$$

for $0 \le t \le 1$, $x \in U$, then $x \in S$; and if one can prove this, the homotopy property for the Leray-Schauder degree implies

$$\deg(I-f\rho,U,0) = \deg(I-fr,U,0) .$$

Because the additivity property gives

$$\deg(I-f\rho,\rho^{-1}(V\cap A),0) = \deg(I-f\rho,U,0) ,$$

and similarly for fr, one then obtains equation (1.18).

Thus it suffices to prove that eq. (1.20) implies $x \in S$. If eq. (1.20) is satisfied, eq. (1.19) implies that

$$x \in f(V\cap A) \subset K_1(f,V) \cap A .$$

In general, assume that

$$x \in K_n(f,V) \cap A .$$

Then one has $\rho(x) = x \in K_n(f,V) \cap A$ and $r(x) \in K_\infty(f,V) \subset A \cap K_n(f,V)$, so

$$(1.21) \qquad (1-t)\rho(x) + tr(x) \in V \cap K_n(f,V) \cap A ,$$

and one concludes from equation (1.21) that

$$x = f((1-t)\rho(x)+tr(x)) \in f(V\cap K_n\cap A) \subset K_{n+1} \cap A .$$

It follows by induction that

$$x \in (\bigcap_{n=1}^{\infty} K_n(f,V)) \cap A \cap V = K_\infty(f,V) \cap V .$$

On $K_\infty \cap V$, equation (1.20) reduces to

$$x \in f(x) ,$$

so $x \in S$ and the proof is complete.

The fixed point index for local strict-set-contractions satisfies generalizations of the additivity, homotopy, commutativity and normalization property. The normalization property involves generalizations of the Lefschetz fixed point theorem which will not be needed in these notes, and I shall omit its statement. The interested reader should look at [95] (for example, Propositions 2.4 and 3.7), [90] (see Theorem 4 on p. 248 and [91]). The additivity property takes the following form:

1. (The additivity property). Assume that X is a closed subset of a Banach space Y, that $X \in F$ and that W is a relatively open subset of X. Suppose that $f : W \to X$ is a continuous map such that $S = \{x\in W | f(x)=x\}$ is compact (possibly empty) and such that there exists a relatively open neighborhood V of S for which $f|V$ is a strict-set-contraction (with respect to some generalized measure of noncompactness β on Y). If W_1 and W_2 are disjoint, relatively open subsets of X (W_1 or W_2 possibly empty) and

$$S \subset W_1 \cup W_2 \subset W ,$$

then,

$$i_X(f,W) = i_X(f,W_1) + i_X(f,W_2) .$$

The following is a version of the homotopy property:

2. (The homotopy property). Assume that X is a closed subset of a Banach space Y, that $X \in F$ and that W is a relatively open subset of X. Suppose that $f : W \times [0,1] \to X$ is a continuous map and that

$$S = \{(x,t) \in W \times [0,1] : f(x,t) = x\}$$

is compact, so $T = \{x \in W : (x,t) \in S \text{ for some } t \in [0,1]\}$ is compact. Finally, assume that there exists a relatively open neighborhood V of T in X, a generalized measure of noncompactness β on Y and a constant c, $0 \leq c < 1$, such that for any set $A \subset V$,

$$\beta(f(A \times [0,1])) \leq c\beta(A) . \tag{1.22}$$

If $f_t(x) = f(x,t)$, one then has

$$i_X(f_0, W) = i_X(f_1, W) . \tag{1.23}$$

In certain cases the homotopy property can be stated more simply. Suppose that X, Y and W are as before and that W is bounded. Assume that $f : \overline{W} \times [0,1] \to X$ is a continuous map such that $f(x,t) \neq x$ for all $(x,t) \in (\overline{W} - W) \times [0,1]$ and such that equation (1.22) is satisfied for all $A \subset \overline{W}$. Then one can prove that T is compact, so equation (1.23) is valid in this situation.

There is also a generalization of the commutativity property to the context of strict-set-contractions:

3. (The commutativity property). Assume that $X_j \in F$, that X_j is a closed subset of a Banach space Y_j and that W_j is a relatively open subset of X_j, $j = 1,2$. Suppose that $f_1 : W_1 \to X_2$ and $f_2 : W_2 \to X_1$ are continuous maps and that $S = \{x \in f_1^{-1}(W_2) : f_2 f_1(x) = x\}$ and $T = f_1(S)$ are compact. Suppose that

β_j is a generalized measure of noncompactness on Y_j , $j = 1,2$, and that there exist relatively open neighborhoods V_1 of S in X_1 and V_2 of T in X_2 , respectively, such that $f_1|V_1$ is a k_1-set-contraction (with respect to β_1 and β_2) and $f_2|V_2$ is a k_2-set-contraction (with respect to β_2 and β_1) . If $k_1k_2 < 1$, one has

$$i_{X_1}(f_2f_1, f_1^{-1}(W_2)) = i_{X_2}(f_1f_2, f_2^{-1}(W_1)) , \tag{1.24}$$

and equation (1.24) is also true if $k_1 = 0$ and f_2 is only continuous.

Section 2

THE KREIN-RUTMAN THEOREM

If X is a real Banach space and C is a closed, convex subset of X, C will be called "a cone (with vertex at 0)" if 1) for all $x \in C$ and for all nonnegative real numbers t, $tx \in C$ and 2) for all $x \in C-\{0\}$, $-x \notin C$. If only the first property holds, C will be called a "wedge". If X^* denotes the continuous, real-valued linear functionals on X and C^* is given by

$$C^* = \{f \in X^* \mid f(x) \geq 0 \text{ for all } x \in C\} ,$$

one can check that C^* is a wedge. If X is the closure of $\{u-v : u,v \in C\}$, C is called "total", and in this case C^* is a cone. Notice that C induces a partial ordering by $x \leq y$ if $y - x \in C$.

If X is a real Banach space and $L : X \to X$ is a bounded linear operator, I shall denote the spectral radius of L by $r(L)$, so

$$(2.1) \qquad r(L) = \lim_{n \to \infty} \|L^n\|^{\frac{1}{n}} .$$

If $\tilde{X} = \{x+iy : x,y \in X\}$ is the complexification of X,

$$\|x+iy\| = \sup_{0 \leq \theta \leq 2\pi} \|(\cos\theta)x+(\sin\theta)y\| ,$$

and $\tilde{L}$ is the obvious linear extension of L to $\tilde{X}$, one can see that $\|\tilde{L}\| = \|L\|$, so

$$r(\tilde{L}) = r(L) \ . \tag{2.2}$$

Equation (2.2) implies that

$$r(L) = \sup\{|z| : z \in \sigma(\tilde{L})\} ,$$

where $\sigma(\tilde{L})$ is the spectrum of $\tilde{L}$.

The Krein-Rutman theorem [76] is a famous and useful result which relates cones to the spectral theory of linear operators:

THEOREM 2.1. (Krein and Rutman [76]). *Suppose that* X *is a real Banach space,* C *is a total cone in* X *and* $L : X \to X$ *is a compact linear operator (i.e., a bounded linear operator which takes bounded sets to precompact sets) such that* $L(C) \subset C$. *If* $r = r(L) > 0$, *there exists* $x_o \in C - \{0\}$ *and* $f_o \in C^* - \{0\}$ *such that*

$$Lx_o = rx_o \quad \textit{and}$$

$$L^* f_o = rf_o \ , \textit{ where } L^* : X^* \to X^* \textit{ is the adjoint of } L \ .$$

In finite dimensions, the condition $r(L) > 0$ is not necessary; but in infinite dimensions, the condition is essential. For example, suppose $X = C[0,1]$, C denotes the nonnegative functions in X and L is defined by

$$(Lx)(t) = \int_0^t x(s)ds \ . \tag{2.3}$$

The operator L is compact and $L(C) \subset C$. However, if

$$(Lx) = \lambda x ,$$

x satisfies

$$x'(t) = \lambda x(t)$$

$$x(0) = 0 ,$$

so x is identically zero. It follows that $r(L) = 0$ (otherwise the Krein-Rutman theorem would imply existence of a positive eigenvector), but L has no eigenvectors.

It has long been recognized that it might be possible to prove the full Krein-Rutman theorem by using fixed point theory. One can easily prove the finite dimensional version of the theorem, the so-called Perron-Frobenius theorem, by an application of the Brouwer fixed point theorem. The original Krein and Rutman article [76] contains theorems concerning eigenvectors of nonlinear, cone-preserving operators, and in fact these theorems were proved by using the Schauder fixed point theorem. However, if the nonlinear results of [76] are specialized to the linear case, they yield a much less general proposition than Theorem 2.1. In his Tata Institute notes on fixed point theory [13], F.F. Bonsall tried to use fixed point theory to obtain Theorem 2.1; but he obtained only partial results. If the cone C has nonempty interior $\overset{\circ}{C}$, $L : X \to X$ is a compact linear operator and $L(C-\{0\}) \subset \overset{\circ}{C}$, P. Rabinowitz [111] used a degree theory argument to prove the existence of an eigenvector in $\overset{\circ}{C}$ with eigenvalue $r = r(L)$. (Note that the weaker assumption that $\overset{\circ}{C}$ is nonempty and $L(\overset{\circ}{C}) \subset \overset{\circ}{C}$ easily implies that $r(L) > 0$.) In Proposition 6 of [103], this author used a fixed point index argument to obtain a nonlinear generalization of Theorem 2.1; however, it was necessary to assume that the cone C is "normal".

Here, I want to present a fixed point index argument which was first given in [107] and which yields a direct generalization of Theorem 2.1 and of an earlier extension by Bonsall [11] of the Krein-Rutman theorem. Before beginning the proof, two points should be made. First, no extraneous assumptions, such as the normality of the cone or nonemptiness of its interior, will be necessary. Second, the fixed point index for mappings defined on relatively open subsets of a closed, convex subset C of a Banach space will be used. As observed in the previous section, the fixed point index can be defined very easily in this case by using a retraction onto C. In fact, all the properties of the fixed point index which will be used in this section are immediate consequences of the corresponding properties of the Leray-Schauder degree. In particular, the argument to be given here can be considered, at least in the case of compact maps, a method for obtaining the Krein-Rutman theorem from the Leray-Schauder degree and the existence of a continuous retraction of the Banach space onto C.

The first lemma is a trivial but useful observation of Bonsall [11]; the proof is left to the reader.

LEMMA 2.1. (Bonsall [11]). *If* $\{a_m : m\geq 1\}$ *is an unbounded sequence of nonnegative reals, there exists a subsequence* $\{a_{m_i} : i\geq 1\}$ *such that* (1) $a_{m_i} \geq i$ *and* (2) $a_{m_i} \geq a_j$ *for* $1 \leq j \leq m_i$.

If C is a cone in Banach space X and $f : D \to X$ is a continuous map, f will be called "order-preserving" (with respect to the partial order induced by C) if whenever $x, y \in D$ and $x \leq y$ one has $f(x) \leq f(y)$. Sometimes the terms "isotonic" or "monotonic" are used instead of "order-preserving". If $f : C \to C$ and $f(tx) = tf(x)$ for all $x \in C$ and all nonnegative reals t, f will be called "positively homogeneous of degree 1". If A is a closed subset of X and U is a relatively open subset of A (so $U = O \cap A$, O open in X), $\partial_A(U)$ will denote the relative boundary of U in A :

$$\partial_A(U) = (\ddot{\bar{U}}-U) .$$

The following lemma is a well-known result (see Corollary 2 on p. 246 of [90]), but I include a proof for completeness.

LEMMA 2.2. *Assume that* D *is a closed, bounded subset of a Banach space* X *and that* $f : D \to X$ *is a strict-set-contraction with respect to a generalized measure of noncompactness* β . *If* A *is any compact subset of* X *and* $S = \{x \in D : x-f(x) \in A\}$, *then* S *is compact (possibly empty). If, for some* $a \in X$, $x - f(x) \neq a$ *for all* $x \in D$, *then there exists* $\delta > 0$ *such that*

$$\inf\{\|x-f(x)-a\| : x \in D\} \geq \delta , \tag{2.4}$$

PROOF. If S and A are as above, then

$$S \subset f(S) + A ,$$

so

$$\beta(S) \leq \beta(f(S)) + \beta(A) = \beta(f(S)) . \tag{2.5}$$

Because f is a strict-set-contraction, equation (2.5) implies that $\beta(S) = 0$, so S has compact closure. Since D is closed and f is continuous, S must be closed, and S itself is compact.

If equation (2.4) is false, select a sequence $\{x_n\}$ such that $\lim_{n\to\infty}\|x_n-f(x_n)-a\| = 0$ and define A by

$$A = \text{closure}\{x_n-f(x_n)-a : n\geq 1\} .$$

By construction A is compact, so the first part of the lemma implies that $\{x_n | n\geq 1\}$ has compact closure. In particular, by taking a subsequence one can assume that $x_n \to x \in D$, and continuity then implies that

$$x = f(x) + a ,$$

a contradiction. □

The first theorem of this section is essentially Theorem 2.1 of [107]; in the case that C is a normal cone, the theorem below is a very special case of an earlier result (Proposition 6 on p. 252 of [103]).

THEOREM 2.2. *Let* C *be a cone in a Banach space* X *and* $f : C \to C$ *a continuous, order-preserving map which is homogeneous of degree 1 and which is a strict-set-contraction with respect to some generalized measure of noncompactness* β *. Assume that there exists* $u \in C$ *such that* $\{\|f^m(u)\| : m \geq 1\}$ *is unbounded, where* f^m *denotes composition of* f *with itself* m *times. If* U *is any relatively open neighborhood of zero in* C *, there exists* $x \in \partial_C(U)$ *and* $t \geq 1$ *such that*

(2.6) $$f(x) = tx .$$

Furthermore, if $f(y) \neq y$ *for all* $y \in C - \{0\}$ *, one has*

(2.7) $$i_C(f,U) = 0 .$$

PROOF. If $f(x) = x$ for $x \in \partial_C(U)$, the theorem is proved, so assume, equivalently, that $f(y) \neq y$ for all nonzero y in C . Suppose that equation (2.7) has been proved but that equation (2.6) is false for all $x \in \partial_C(U)$ and $t \geq 1$. Consider the homotopy

(2.8) $$f_s(x) = sf(x)$$

for $0 \leq s \leq 1$ and $x \in \overline{U}$. By assumption, $f_s(x) \neq x$ for $x \in \partial_C(U)$ and $0 \leq s \leq 1$. If $S = \{(x,s) \in \overline{U} \times [0,1] : f_s(x) = x\}$ and

$T = \{x\in\overline{U} : (x,s)\in S$ for some $s\in[0,1]\}$, equation (2.8) implies that

$$T \subset \overline{co}\{f(T)\cup\{0\}\} ,$$

so

$$\beta(T) \leq \beta(f(T)) ,$$

and the fact that f is a strict-set-contraction implies that $\beta(T) = 0$ and T is compact. The other hypotheses of the homotopy property are easily verified and one concludes that

$$i_C(f_1,U) = i_C(f,U) = i_C(f_o,U) . \tag{2.9}$$

Because f_o is the map which takes all points to zero, the definition of the fixed point index implies

$$i_C(f_o,U) = \deg(I,W,0) = 1 , \tag{2.10}$$

where W is an open neighborhood of zero in X . Alternatively, one can argue that the commutativity and normalization properties of the fixed point index imply that $i_C(f_o,U)$ equals the Lefschetz number of the identity map on a single point space, and hence is 1. In any event one concludes that

$$i_C(f,U) = 1 ,$$

which contradicts equation (2.7).

It remains to prove equation (2.7). By Lemma 2.2, there exists $\delta > 0$ such that

$$\inf\{\|x-f(x)\| : x\in\partial_C(U)\} \geq \delta .$$

Because f is homogeneous of degree 1 one can, by multiplying u by a positive constant, assume that

$$\|u\| < \delta .$$

Define $g(x) = f(x) + u$ and consider the homotypy $f(x) + su$, $0 \le s \le 1$. The hypotheses of the homotopy property are easily verified, and one concludes that

$$i_C(f,U) = i_C(g,U) . \tag{2.11}$$

To complete the proof, it suffices to prove that $g(x) \ne x$ for all $x \in U$.

If $g(x) = x$ for some $x \in U$, one obtains

$$x \ge u .$$

Assuming, by way of mathematical induction, that

$$x \ge f^m(u) , \tag{2.12}$$

one finds

$$x = g(x) = f(x) + u \ge f(x) \ge f(f^m(u))$$
$$= f^{m+1}(u) ,$$

so equation (2.12) is true for all m.

Define $a_m = \|f^m(u)\|$ and let a_{m_i} be a subsequence as in Lemma 2.1. If $v_m \overset{\text{def}}{=} \dfrac{f^m(u)}{\|f^m(u)\|}$ and $S = \{v_{m_i} : i \ge 1\}$, I claim that S has compact closure. Assuming this to be true for the moment, then by taking a subsequence $v_{m_{i_j}} \overset{\text{def}}{=} w_j$ one can assume that $w_j \to w \in C$ and $\|w\| = 1$. However, equation (2.12) gives

$$(a_{m_{i_j}})^{-1}x - w_j \in C ,$$

and by taking limits as $j \to \infty$, one obtains the contradiction that $-w \in C$.

Thus to complete the proof it suffices to prove that $\beta(S) = 0$. Notice that one can write

$$S = (\bigcup_{i=1}^{k} \{v_{m_i}\}) \cup f^k(T_k) , \text{ where}$$

(2.13)

$$T_k \overset{\text{def}}{=} \{\frac{f^{m_i - k}(u)}{a_{m_i}} : i>k\}$$

The definition of the subsequence a_{m_i} implies that

(2.14) $$T_k \subset \{x \in C : \|x\| \le 1\} \overset{\text{def}}{=} B .$$

Since f is a c-set-contraction, $c < 1$, with respect to β, equation (2.13) and (2.14) give

(2.15) $$\beta(S) = \beta(f^k(T_k)) \le c^k \beta(B) .$$

The right hand side of equation (2.15) approaches 0 as $k \to \infty$, so $\beta(S) = 0$. □

The remaining results of this section, including the Krein-Rutman theorem, will all be obtained as corollaries of Theorem 2.2. Before giving the first corollary it is convenient to introduce a definition.

DEFINITION 2.1. A generalized measure of noncompactness β on a Banach space X will be called *positively homogeneous of degree* 1 if

$$\beta(tA) = t\beta(A)$$

for all bounded sets $A \subset X$ and all reals $t \ge 0$.

One can easily see that the Kuratowski's measure of noncompactness (see equation 1.13, Section 1) is positively homogeneous of degree 1.

COROLLARY 2.1. *Let* C *be a cone in a Banach space* X *and* β *a generalized measure of noncompactness on* X, β *positively homogeneous of degree 1. Let* $g : C \to C$ *be an order-preserving map which is positively homogeneous of degree 1 and which is a c-set-contraction w.r.t.* β. *In addition suppose that there exists* $u \in C-\{0\}$, $a > c$ *and an integer* $p \geq 1$ *such that*

$$g^p(u) \geq a^p u .$$

If W *is any relatively open neighborhood of* 0 *in* C, *there exists* $x_o \in \partial_C(W)$ *and* $\lambda \geq a$ *such that*

$$g(x_o) = \lambda x_o .$$

PROOF. Let $\{a_n\}$ be a monotone increasing sequence such that $c < a_n < a$ and $\lim_{n\to\infty} a_n = a$. Suppose that for each $n \geq 1$ there exists $\lambda_n \geq a_n$ and $x_n \in \partial_C(W)$ such that

$$g(x_n) = \lambda_n x_n .$$

Let $S = \{x_n | n\geq 1\}$; if one can prove that S has compact closure, i.e., $\beta(S) = 0$, then by taking a convergent subsequence $x_{n_i} \to x$ and $\lambda_{n_i} \to \lambda$, one completes the proof. However,

$$\beta(S) = \beta(\{(\frac{1}{\lambda_n})g(x_n) : n\geq 1\}) ,$$

and because

$$\{(\frac{1}{\lambda_n})g(x_n) : n\geq 1\} \subset \overline{co}(\{\frac{1}{a_1} g(x_n) : n\geq 1\} \cup \{0\}) ,$$

one concludes that

$$\beta(S) \le \beta(\{\frac{1}{a_1} g(x_n) : n\ge 1\}) = (\frac{1}{a_1})\beta(g(S))$$

$$\le (\frac{c}{a_1})\beta(S) .$$

Because $(\frac{c}{a_1}) < 1$, the previous inequality implies $\beta(S) = 0$.

The above remarks show that it suffices to prove that for each real number b , $c < b < a$, there exists $x \in \partial_C(W)$ and $\lambda \ge b$ such that $g(x) = \lambda x$. Define $f(x) = (\frac{1}{b})g(x)$. A simple induction implies

$$f^{jp}(u) \ge (\frac{a}{b})^{jp}u$$

for all $j \ge 1$, so

$$(\frac{b}{a})^{jp}f^{jp}(u) - u \in C \quad , \quad j \ge 1 .$$

If $\|f^{jp}(u)\|$ were bounded, the previous equation would imply (take limits as $j \to \infty$) that $-u \in C$, a contradiction. Thus $\{f^m(u) : m\ge 1\}$ is unbounded and, of course, f is a $(\frac{c}{b})$-set-contraction w.r.t. β . Theorem 2.2 implies that there exists $x \in \partial_C(W)$ and $t \ge 1$ so

$$f(x) = tx \quad \text{or} \quad g(x) = (tb)x ,$$

which completes the proof. □

The essential point in Theorem 2.2 is that there exists a relatively open neighborhood W of 0 in C and a continuous map $f : W \to C$ such that $i_C(f,W) = 0$. The assumptions on f in Theorem 2.2 are only important in that they imply that $i_C(f,W) = 0$, and we shall see in the next section that other reasonable hypotheses lead to the same conclusion. The next two corollaries illustrate this point by weakening the order-preserving and homogeneity assumptions in Theorem 2.2.

COROLLARY 2.2. *Let assumptions and notation be as in Theorem 2.2 and suppose that* $f(x) \neq x$ *for all* $x \in C - \{0\}$. *For* $r > 0$, *define* $V_r = \{x \in C : \|x\| \leq r\}$ *and suppose that for* $r_0 > 0$, $h : V_{r_0} \to C$ *is a strict-set-contraction w.r.t.* β *such that*

$$\lim_{r \to 0^+} \sup\{\frac{\|f(x)-h(x)\|}{\|x\|} : 0<\|x\|<r\} = 0 .$$

There exists $r_1 > 0$ *such that if* W *is any relatively open neighborhood of* 0 *in* C *with* $W \subset V_{r_1}$, *then*

$$i_C(h,W) = 0 ,$$

and there exists $x \in \partial_C(W)$ *and* $t > 1$ *such that* $h(x) = tx$.

PROOF. Lemma 2.2 implies that there exists $\delta > 0$ such that

$$\|f(x)-x\| \geq \delta \quad , \quad \|x\| = 1 .$$

Homogeneity implies that

$$\|f(x)-x\| \geq \delta\|x\| \tag{2.16}$$

for all $x \in C$; and if $r_1 > 0$ is chosen so that

$$\|f(x)-h(x)\| < \delta\|x\| \quad \text{for} \quad 0 < \|x\| < r_1 , \tag{2.17}$$

one can easily show that for $0 \leq s \leq 1$ and $0 < \|x\| \leq r_1$,

$$\|(1-s)f(x)+sh(x)-x\| > 0 .$$

The homotopy property for the fixed point index and Theorem 2.2 now imply that if W is a relatively open neighborhood of 0 in C, $W \subset V_{r_1}$,

$$i_C(h,W) = i_C(f,W) = 0 .$$

If $h(x) \neq tx$ for some $t > 1$ and $x \in \partial_C(W)$, then just as in the proof of Theorem 2.2, the homotopy $sh(x)$, $0 \le s \le 1$, would imply that $i_C(f,W) = 1$, a contradiction. □

A technical lemma is needed before proving the next corollary.

LEMMA 2.3. *Let* C *be a cone in a Banach space* X *and* β *a generalized measure of noncompactness on* X *such that* β *is positively homogeneous of degree 1. For some* $R \ge 0$, *assume that* $h : \{x \in C : \|x\| \ge R\} \to C$ *is a c-set-contraction w.r.t.* β. *If* $H : C \to C$ *is defined by*

$$H(x) = \begin{cases} h(x) & \text{for } \|x\| \ge R \\ \frac{\|x\|}{R} h\left(\frac{Rx}{\|x\|}\right) & \text{for } 0 < \|x\| \le R \\ 0 & \text{for } x = 0 . \end{cases}$$

H *is continuous and is a c-set-contraction w.r.t.* β.

PROOF. The proof of continuity is easy and is left to the reader. Select a number $c_1 > c$; it suffices to prove that for every bounded set

$$\beta(H(A)) \le c_1 \beta(A) . \tag{2.18}$$

If $\bar{A}$ is compact, $\overline{H(A)}$ is compact and equation (2.18) is immediate, so assume $\beta(A) > 0$. For $0 \le r < s \le \infty$, define $A_{r,s}$ by

$$A_{r,s} = \{x \in A : r \le \|x\| \le s\} .$$

Because $\beta(A) > 0$, there exists $\varepsilon > 0$ such that

$$\beta(H(A_{o,\varepsilon})) \le c\beta(A) ,$$

and because h is a c-set-contraction,

$$\beta(H(A_{R,\infty})) = \beta(h(A_{R,\infty})) \le c\beta(A_{R,\infty}) \le c\beta(A) .$$

Select $\delta > 0$ so that $(1+\delta)c \le c_1$. One can write A as the union of $A_{o,\varepsilon}$, $A_{R,\infty}$ and a finite number of sets of the form $A_{r,s}$, where $\varepsilon \le r < s \le \infty$ and $s \le (1+\delta)r$, so $\beta(H(A))$ is the maximum of $\beta(H(A_{o,\varepsilon}))$, $\beta(H(A_{R,\infty}))$ and $\beta(H(A_{r,s}))$ (r,s as above). Thus it suffices to prove that

(2.19) $$\beta(A_{r,s}) \le c_1\beta(A)$$

for r and s as above. To prove equation (2.19) note that

$$H(A_{r,s}) \subset \overline{co}(\{\frac{s}{R}h(\frac{Rx}{\|x\|}) : x\epsilon A_{r,s}\}\cup\{0\}) ,$$

so

(2.20) $$\beta(H(A_{r,s})) \le \frac{s}{R}\beta(\{h(\frac{Rx}{\|x\|}) : x\epsilon A_{r,s}\}) .$$

Because one has

(2.21) $$\{\frac{Rx}{\|x\|}: x\epsilon A_{r,s}\} \subset \overline{co}(\{\frac{R}{r}x : x\epsilon A_{r,s}\}\cup\{0\} ,$$

and because h is a c-set-contraction, equation (2.20) implies

(2.22) $$\beta(H(A_{r,s})) \le (\frac{s}{R})(c)(\frac{R}{r})\beta(A_{r,s}) = (\frac{cs}{r})\beta(A_{r,s}) \le c_1\beta(A) ,$$

which is the desired estimate. □

Lemma 2.3 illustrates one of the technical difficulties of working with c-set-contractions: given a c-set-contraction h it is not always clear that h has a suitable extension which is also a c-set-contraction.

COROLLARY 2.3. *Let assumptions and notation be as in Theorem 2.2 and assume that* $f(x) \neq x$ *for all* $x \in C - \{0\}$. *Assume that for some* $R > 0$, $h : \{x \in C : \|x\| \geq R\} \to C$ *is a strict-set-contraction w.r.t.* β *and that*

$$\lim_{r\to\infty} \sup\{\frac{\|f(x)-h(x)\|}{\|x\|} : \|x\| \geq r, x \in C\} = 0 .$$

Then there exists a number $R_1 \geq R$ *such that if* W *is any bounded relatively open neighborhood of* 0 *with* $\partial_C(W) \subset \{x \in C : \|x\| \geq R_1\}$, $i_C(h,W) = 0$ *and there exist* $x \in \partial_C(W)$ *and* $t > 1$ *with* $h(x) = tx$.

PROOF. By using Lemma 2.3 one can assume that $h : C \to C$ is a strict-set-contraction w.r.t. β . Just as in Corollary 2.2, there exists $\delta > 0$ such that

$$\|x-f(x)\| \geq \delta \|x\| ,$$

and if $R_1 > R$ is chosen so that

$$\|f(x)-h(x)\| < \delta\|x\| \quad \text{for} \quad \|x\| > R_1 ,$$

the homotopy $(1-s)f(x) + sh(x)$ shows that

$$i_C(h,W) = i_C(f,W) = 0$$

whenever $\partial_C(W) \subset \{x \in C : \|x\| \geq R_1\}$ and W is a bounded, relatively open neighborhood of 0 . The final statement of the corollary follows just as in Corollary 2.2. □

I next want to show how Theorem 2.1 can be used to obtain various generalizations of the linear Krein-Rutman theorem. First, some definitions will be useful. If S is a bounded subset of a real Banach space X, $\alpha(S)$ will always denote the Kuratowski measure of noncompactness of S:

$$\alpha(S) = \inf\{d>0 \mid S \text{ is a finite union of sets of diameter } \delta \le d\} .$$

If $L : X \to X$ is a bounded linear operator,

$$\alpha(L) = \inf\{c>0 \mid L \text{ is a c-set-contraction}\} , \tag{2.23}$$

and we define $\rho(L)$ by

$$\rho(L) = \lim_{n\to\infty}(\alpha(L^n))^{\frac{1}{n}} = \inf_{n\ge 1} \alpha(L^n)^{\frac{1}{n}} \tag{2.24}$$

It is proved in [106] that $\rho(L)$ is the so-called "essential spectral radius" of L and has a natural interpretation in terms of the spectrum of the complexification of L. (One can prove that the limit in eq. (2.24) exists by using the fact that $0 \le \alpha(L_1L_2) \le \alpha(L_1)\alpha(L_2)$.)

If K is a cone in X and $L : X \to X$ is a bounded linear operator such that $L(K) \subset K$, one can also speak of the cone spectral radius, $r_K(L)$, and the cone essential spectral radius, $\rho_K(L)$. More precisely, if $L(K) \subset K$, define

$$\|L\|_K = \sup\{\|Lx\| : x \in K \text{ and } \|x\| \le 1\}$$

and

$$r_K(L) = \lim_{n\to\infty}\|L^n\|_K^{\frac{1}{n}} = \inf_{n>o} \|L^n\|_K^{\frac{1}{n}} . \tag{2.25}$$

Define $\alpha_K(L)$ by

$$\alpha_K(L) = \inf\{c>0 : L|K \text{ is a c-set-contraction}\}$$

and

(2.26)
$$\rho_K(L) = \lim_{n\to\infty}(\alpha_K(L^n))^{\frac{1}{n}} = \inf_{n>0}(\alpha_K(L^n))^{\frac{1}{n}} .$$

One can easily see that $\rho(L) \le r(L)$, $r_K(L) \le r(L)$ and $\rho_K(L) \le \rho(L)$. If the cone K is "reproducing" (so $X=\{u-v : u,v\in K\}$) one can prove that

(2.27)
$$\rho_K(L) \le r_K(L) ,$$

but the proof will be omitted here. I do not know whether inequality (2.27) is true for all L without some restriction on K . Notice that $\alpha(L) = 0$ if and only if L is compact and $\alpha_K(L) = 0$ if and only if $L|K$ is compact.

If K is a cone in a Banach space X with norm $\|\cdot\|$, define a vector space Y by

$$Y = \{u-v : u,v\in K\}$$

and make Y into a normed linear space by

(2.28)
$$|y| \overset{\text{def}}{=} \inf\{\|u\|+\|v\| : y=u-v, u,v\in K\} .$$

It is clear that $|y| \ge \|y\|$ for all $y \in Y$ and $|y| = \|y\|$ if $y \in K$. One can also easily check that if L is a linear operator whose domain contains K , and L satisfies $L(K) \subset K$ and $\|L\|_K < \infty$, then L induces a bounded linear operator on Y and one has

(2.29)
$$|L|_Y = \|L\|_K \text{ and } r_Y(L) = r_k(L) ,$$

where $|L|_Y$ is the norm of $L : Y \to Y$ and $r_Y(L)$ its spectral radius.

It is well-known that $(Y,|\cdot|)$ is, in fact, a Banach space: see, for example, [116]. For completeness, I sketch a proof.

LEMMA 2.4. $(Y,|\cdot|)$ *is a complete normed linear space.*

PROOF. Suppose $y_n \in Y$, $n \geq 1$, and $\sum_{n=1}^{\infty} |y_n| < \infty$. To prove the lemma, one must find $y \in Y$ such that

$$\lim_{N\to\infty} |y - \sum_{j=1}^{N} y_j| = 0 . \tag{2.30}$$

By definition, there exist $u_n, v_n \in K$ such that $y_n = u_n - v_n$ and $\|u_n\| + \|v_n\| \leq 2|y_n|$. It follows that

$$\sum_{n=1}^{\infty} \|u_n\| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \|v_n\| < \infty ,$$

and the completeness of $(X,\|\cdot\|)$ implies that there exist $u \in K$ and $v \in K$ such that

$$\sum_{n=1}^{\infty} u_n = u \quad \text{and} \quad \sum_{n=1}^{\infty} v_n = v , \tag{2.31}$$

where the convergence of the infinite sums in (2.31) is in the $\|\cdot\|$ norm. If one defines $y = u - v$, one can easily verify equation (2.30). □

COROLLARY 2.4. *If* X *is a real Banach space,* K *a cone in* X *and* $L : X \to X$ *a bounded linear operator such that* $L(K) \subset K$, *define* $b = \rho_K(L)$ *and* $a = r_K(L)$ *(where* $\rho_K(L)$ *and* $r_k(L)$ *are defined in equations (2.25) and (2.26)), and assume that* $b < a$. *Then it follows that there exists* $x \in K - \{0\}$ *such that* $Lx = ax$.

PROOF. Suppose one can find a sequence of real numbers μ_n, $b < \mu_n$, $\lim_{n\to\infty} \mu_n = a$ and vectors $x_n \in K$, $\|x_n\| = 1$, such that

$$Lx_n = \mu_n x_n . \tag{2.32}$$

Notice that if μ is an eigenvalue of L with eigenvector in K, then the definition of a implies that $\mu \le a$, so automatically $\mu_n \le a$. Select $d > b$ such that $\mu_n \ge d$ for all n and define $S = \{x_n : n\ge1\}$. It suffices to prove that $\alpha(S) = 0$, because then there exists a convergent subsequence $\{x_{n_i}\}$, converging to $x \in K$, $\|x\| = 1$, and equation (2.32) implies $Lx = ax$. However, if $V = \{x\in K : \|x\|\le1\}$, it is easy to see that for any $j \ge 1$,

$$S = \{(\frac{L}{\mu_n})^j(x_n) : n\ge1\} \subset (\frac{L}{d})^j(V) . \tag{2.33}$$

The definition of $\rho_K(L)$ implies that

$$\alpha((\frac{L}{d})^j(V)) = (\frac{1}{d})^j\alpha(L^j(V)) \to 0 \quad \text{as} \quad j \to \infty ,$$

so $\alpha(S) = 0$.

The previous remarks show that it suffices to prove that for every real number s, $b < s < a$, there exists $x \in K$, $\|x\| = 1$, and $\mu \ge s$ such that

$$Lx = \mu x .$$

Define $g(x) = (\frac{1}{s})(Lx)$, so

$$\rho_K(g) = (\frac{b}{s}) < 1 < (\frac{a}{s}) = r_K(g) .$$

Select an integer N so that $\alpha_K(g^m) \le c < 1$ for $m \ge N$, and as in [107] define a generalized measure of noncompactness β by

$$\beta(A) = (\frac{1}{N}) \sum_{j=0}^{N-1} \alpha(g^j(A)) . \tag{2.34}$$

One can easily check that β is indeed a generalized measure of noncompactness which is positively homogeneous of degree 1. Moreover, one has

$$\beta(g(A)) = \frac{1}{N}\sum_{j=1}^{N-1}\alpha(g^j(A)) + \frac{1}{N}\alpha(g^N(A))$$

(2.35)

$$\leq (\frac{1}{N})c\alpha(A) + \frac{1}{N}\sum_{j=1}^{N-1}\alpha(g^j(A))$$

If c_1 is chosen so that $c < c_1 < 1$ and

(2.36) $$(1-c_1)\sum_{j=1}^{N-1}\|g^j\| \leq (c_1-c) ,$$

one can obtain from equation (2.35) that

$$\beta(g(A)) \leq c_1\beta(A) ,$$

so g is a strict-set-contraction w.r.t. β .

To complete the proof, it suffices to prove that g has an eigenvector $x \in K$ with eigenvalue $t \geq 1$, and this will follow from Theorem 2.2 if there exists $u \in K$ such that $\{\|g^m(u)\| : m\geq 1\}$ is unbounded. If Y is as in Lemma 2.4, one knows that $r_K(g) = r_Y(g) > 1$, so $|g^m|_Y$ is unbounded. The uniform boundedness principle implies that there exists $y \in Y$ such that $\{|g^m(y)|_Y : m\geq 1\}$ is unbounded. If $y = v - w$, with $v,w \in K$, it must be true that $\|g^m(v)\|$ or $\|g^m(w)\|$ is unbounded, so there exists $u \in K$ with $\{\|g^m(u)\| : m\geq 1\}$ unbounded. □

REMARK 2.1. If $\rho_K(L) = 0$, so $L|K$ is compact, Corollary 2.4 generalizes a theorem of F. Bonsall [11]. As Bonsall points out [11], his theorem is <u>not</u> a special case of the Krein-Rutman theorem, even if the cone K is total. It may be worthwhile to describe Bonsall's striking example. Take $X = \{\varphi\in C[0,1] : \varphi(0)=0\}$ in the usual sup norm and define $L : X \to X$ by $(Lx)(t) = x(\frac{t}{2})$. Define $K = \{\varphi\in X : \varphi$ is convex <u>and</u> monotone increasing$\}$ (by a function φ being monotone increasing on an interval J we shall always mean $\varphi(s)\leq\varphi(t)$ for all $s<t$ in J) and define, for $0 < \gamma \leq 1$, a linear homeomorphism $S_\gamma : X \to X$ by $(S_\gamma x)(t) = x(t^\gamma)$. Define K_γ, $0 < \gamma \le 1$, to be the cone $S_\gamma(K)$. Bonsall observes that $L(K_\gamma) \subset K_\gamma$ and that $L|K_\gamma$ is compact. If $x_\gamma(t) = t^\gamma$, $x_\gamma \in K_\gamma$ and $L(x_\gamma) = (\frac{1}{2})^\gamma x_\gamma$, so $r_{K_\gamma}(L) \ge (\frac{1}{2})^\gamma$; Bonsall proves that in fact $r_{K_\gamma}(L) = (\frac{1}{2})^\gamma$. Nevertheless, one can prove that K is total, so K_γ is total. Finally, one can prove that $L : X \to X$ is not compact. In fact it will follow from the next lemma that $\rho(L) = r(L) = 1$.

It is interesting to note that linear maps of the same general type, but much more complicated, have arisen in work of Bumby [24] on the Hausdorff dimension of sets of real numbers. Some discussion of the spectrum of such maps is given in Theorem 2.3 in [107].

The final theorem of this section will be a direct generalization of the linear Krein-Rutman theorem, but another lemma is needed first. In the case that L is a compact linear map, the following lemma was first proved by F.F. Bonsall [11].

LEMMA 2.5. (Compare [11]). *Let* C *be a total cone in a real Banach space* X *and* $L : X \to X$ *a bounded linear map such that* $\rho(L) < r(L)$, *where* $\rho(L)$ *is the essential spectral radius of* L *(given by equation (22.4)) and* $r(L)$ *is the spectral radius of* L. *If* $L(C) \subset C$, *one has*

$$r_C(L) = r(L) ,$$

where $r_C(L)$ *is the cone spectral radius of* L *(equation (2.25)).*

PROOF. Suppose one can prove the existence of $x \in X$ such that

$$\limsup_{n \to \infty} \frac{\|L^n x\|}{\|L^n\|} = \delta > 0 . \qquad (2.37)$$

Because C is total, there exists $y = v - w$, $v,w \in C$, such that $\|x-y\| < \frac{\delta}{2}$, and it follows that

$$\limsup_{n \to \infty} \frac{\|L^n y\|}{\|L^n\|} \geq \frac{\delta}{2} > 0 . \tag{2.38}$$

Equation (2.38) implies that

$$\limsup_{n \to \infty} \frac{\|L^n u\|}{\|L^n\|} > 0 , \tag{2.39}$$

where $u \in C$ and $u = v$ or $u = w$. Equation (2.39) implies that

$$r_C(L) \geq \limsup_{n \to \infty} \|L^n u\|^{\frac{1}{n}} \geq r(L) ,$$

and because one always has $r_C(L) \leq r(L)$, the lemma is proved.

Thus it suffices to find x as in eq. (2.37). By way of contradiction assume that for all x

$$\lim_{n\to\infty} \frac{\|L^n x\|}{\|L^n\|} = 0 . \tag{2.40}$$

Select numbers ρ_1 and ρ_2 such that

$$\rho(L) < \rho_1 < \rho_2 < r(L) . \tag{2.41}$$

If $B = \{x \in X : \|x\| \leq 1\}$, there exists an integer N so that

$$\alpha(L^n(B)) < \rho_1^n \alpha(B) \leq 2\rho_1^n \quad \text{for all} \quad n \geq N , \tag{2.42}$$

and N can also be chosen so large that

$$2^{\frac{1}{N}} \rho_2 < r(L) .$$

By the definition of α , there exist sets $S_1, S_2, \ldots, S_m$ such that

$$L^N(B) = \bigcup_{j=1}^{m} S_j \quad \text{and}$$

$$\operatorname{diam}(S_j) < 2\rho_1^N \quad \text{for} \quad 1 \le j \le m \,.$$

For $1 \le j \le m$, select $x_j \in S_j$ and given $\varepsilon > 0$ such that

$$\varepsilon < 2\rho_2^N - 2\rho_1^N \,,$$

select (by using equation (2.40)) an integer N_1 such that

$$\frac{\|L^n x_j\|}{\|L^n\|} < \varepsilon \quad \text{for} \quad n \ge N_1 \quad \text{and} \quad 1 \le j \le m \,.$$

Given $x \in B$, select j such that $L^N(x) \in S_j$. For any $n \ge N_1$ one has

$$\begin{aligned} \|L^{n+N}(x)\| &\le \|L^n(L^N(x)-x_j)\| + \|L^n x_j\| \\ &\le \|L^n\|(2\rho_1^N) + \varepsilon\|L^n\| \qquad (2.43) \\ &\le 2\rho_2^N\|L^N\| \;, \end{aligned}$$

so one obtains

$$\|L^{n+N}\| \le 2\rho_2^N\|L^n\| \quad \text{for} \quad n \ge N_1 \,. \qquad (2.44)$$

If k_1 is selected so that $k_1 N \ge N_1$ and one applies equation (2.44) repeatedly one obtains

$$\begin{aligned} \|L^{kN}\| &\le (2\rho_2^N)^{k-k_1}\|L^{k_1 N}\| \qquad (2.45) \\ &\le D(2\rho_2^N)^k \;, \end{aligned}$$

where D is a constant independent of k. It follows that

$$(2.46) \qquad \lim_{k\to\infty} \|L^{kN}\|^{(\frac{1}{kN})} \le 2^{\frac{1}{N}} \rho_2 < r(L) ,$$

which is a contradiction. □

As a final corollary of Theorem 2.2 I shall now derive a direct generalization of the linear Krein-Rutman theorem (Theorem 2.1), to which the following result reduces when L is compact.

COROLLARY 2.5. *Let* X *be a real Banach space and* $L : X \to X$ *a bounded linear map such that* $\rho(L) < r(L)$ *, where* $\rho(L)$ *is the essential spectral radius of* L *(equation (2.24) and* $r(L)$ *the spectral radius (equation (2.1)). If* C *is any total cone in* X *such that* $L(C) \subset C$ *, one has*

$$(2.47) \qquad r_C(L) = r(L) ,$$

and there exists $x_o \in C - \{0\}$ *such that*

$$(2.48) \qquad Lx_o = rx_o \, , \; r = r(L) .$$

If X^* *is the dual space of* X *and* $C^* = \{f\in X^* : f(x)\ge 0$ for all $x\in C\}$ *, there exists* $f_o \in C^* - \{0\}$ *such that*

$$(2.49) \qquad L^* f_o = rf_o .$$

PROOF. Lemma 2.5 implies equation (2.47). Because one always has $\rho_C(L) \le \rho(L)$, it follows that $\rho_C(L) < r_C(L) = r$, and Corollary 2.4 gives equation (2.48).

It is proved in [106] that $\rho(L^*) = \rho(L)$ (this is a classical result if L is compact), so $\rho_{C^*}(L^*) \le \rho(L)$. If one can prove that

(2.50) $$r_{C^*}(L^*) \geq r(L) = r ,$$

the existence of f_o as in equation (2.49) again follows from Corollary 2.4. However, because $-x_o \notin C$, the Hahn-Banach theorem implies that there exists $g \in C^*$ such that $\langle x_o, g\rangle > 0$ (where $\langle , \rangle$ denotes the bilinear pairing between X and X^*). Furthermore, one has

(2.51) $$\langle x_o, (L^*)^n g\rangle = \langle L^n x_o, g\rangle = r^n \langle x_o, g\rangle ,$$

and equation (2.51) implies equation (2.50). □

Section 3

THE mod p THEOREM FOR THE FIXED POINT INDEX

I wou like to begin this section by describing a useful theorem, sometimes called the "mod p theorem", which relates the fixed point index of f^m to that of f when m is a prime p or a power of p. This result was obtained independently by Steinlein [118,119] and Krasnosel'skii and Zabreiko [75] in the early seventies, though Kranosel'skii and Zabreiko did not present a complete proof in [75]. In the case of maps of $\mathbb{R}^n$ into $\mathbb{R}^n$, the mod p theorem takes the following form:

THEOREM 3.1. (Steinlein, Krasnosel'skii and Zabreiko). *Suppose that* G *is an open subset of* $\mathbb{R}^n$, $f : G \to \mathbb{R}^n$ *is a continuous map and* H *is an open subset of* G *such that* f^m *is defined on* H, *where* $m = p^t$ *and* p *is a prime. Assume that* $\Sigma = \{x \in H : f^m(x)=x\}$ *is compact and that* $f(\Sigma) \subset \Sigma$. *Then one has*

$$\deg(I-f,H,0) \equiv \deg(I-f^m,H,0) \quad . \quad (\text{mod } p) \ .$$

If one writes $X = \mathbb{R}^n$, the conclusion of Theorem 3.1 can be restated in the notation of the fixed point index:

$$i_X(f,H) \equiv i_X(f^m,H) \quad (\text{mod } p) \ .$$

Indeed, it is not too hard to extend Theorem 3.1 to the context of compact maps or strict-set-contractions on ANR's. For example, Steinlein [119] has proven:

THEOREM 3.2. (Steinlein). *Suppose that* G *is an open subset of a metric ANR* X, $f : G \to X$ *is a continuous map and* H *is an open subset of* G *such that* f^m *is defined on* H, *where* $m = p^t$ *and* p *is a prime. Assume that* $\Sigma = \{x \in H : f^m(x) = x\}$ *is compact (possibly empty), that* $f(\Sigma) \subset \Sigma$ *and that* f *is compact on some neighborhood of* Σ. *Then one has*

$$i_X(f^m, H) \equiv i_X(f, H) \pmod p .$$

I shall now present a proof of Theorem 3.1. The general approach is to show that one can approximate f by "nicer" maps for which the theorem is easier to prove. At the final stage one reduces to the case that $m = p$, f is a C^∞ map on H and $f(x) \neq x$ for all $x \in \overline{H}$ (Lemma 3.4). At this point one can fill in the details of a remark of Tromba (see [123], p. 488) to find an approximation of f for which the theorem is easy to prove. Because the fixed point index is stable under small perturbations, one obtains the desired result for the original f by taking close enough nice approximations.

Theorem 3.2 can, in fact, be derived from Theorem 3.1: see [119]. In the interests of length and because all the essential difficulties already arise in the proof of Theorem 3.1, I shall omit the proof, but I shall use Theorem 3.2 later.

The first step toward proving Theorem 3.1 is to prove it when f is linear. First, some notation is needed. If $A : \mathbf{R}^n \to \mathbf{R}^n$ is a linear map (an $n \times n$ real matrix), A extends to a complex linear map $A : \mathbb{C}^n \to \mathbb{C}^n$. If $z \in \mathbb{C}$, define an integer $n_z(A)$ by

(3.1) $$n_z(A) = \dim\{w \in \mathbb{C}^n : (I - zA)^j(w) = 0 \text{ for some } j \geq 1\} ,$$

where "dim" in equation (3.1) is complex dimension. A result of Leray implies that if $I - A$ is one-one and U is any open neighborhood of 0 in $\mathbf{R}^n$, then

$$\deg(I-A,U,0) = (-1)^N \text{, where} \tag{3.2}$$

$$N = \sum_{0<t<1} n_t(A) \ . \tag{3.3}$$

Note that for t real,

$$n_t(A) = \dim\{w \in \mathbf{R}^n : (I-tA)^j w=0 \quad \text{for some} \quad j \geq 1\} \ , \tag{3.4}$$

where "dim" in equation (3.4) is the dimension of a real subspace of $\mathbf{R}^n$. Actually, Leray proved equation (3.2) in the case that X is a real Banach space, $A : X \to X$ is a compact linear map and U is an open neighborhood of 0. Equation (3.2) also makes sense and is valid if $A : X \to X$ is a bounded linear operator whose essential spectral radius is less then one.

With these preliminaries, the first lemma can be proved.

LEMMA 3.1. *Assume that* $A : \mathbf{R}^n \to \mathbf{R}^n$ *is a linear map and that for some positive integer* m, $I - A^m$ *is one-one (so necessarily* $I-A$ *is one-one). If* m *is an odd integer and* U *is an open neighborhood of* 0, *then*

$$\deg(I-A^m,U,0) = \deg(I-A,U,0) \ . \tag{3.5}$$

If $m = 2^t$ *for some integer* t, *then*

$$\deg(I-A^m,U,0) \equiv \deg(I-A,U,0) \pmod 2 \ . \tag{3.6}$$

PROOF. If $L : U \to \mathbf{R}^n$ is a nonsingular linear map, it is well-known that $\deg(L,U,0) = \pm 1$, so equation (3.6) follows from the trivial fact that $+1$ and -1 are congruent mod 2.

Thus assume m is odd. There exists a linear map $B : \mathbf{R}^n \to \mathbf{R}^n$ so close to A that

$$\deg(I-A,U,0) = \deg(I-B,U,0)$$

and

$$\deg(I-A^m,U,0) = \deg(I-B^m,U,0) \ ,$$

and such that all eigenvalues of B are algebraically simple. Thus one can assume from the start that all eigenvalues of A are algebraically simple. Equations (3.2) and (3.3) show that the lemma will follow if one can prove that

$$n_t(A) \equiv n_{t^m}(A^m) \pmod 2 \quad \text{for} \quad 0 < t < 1 \ . \tag{3.7}$$

A simple calculation shows that

$$I - t^m A^m = \prod_{j=0}^{m-1} (I-\xi_j A) \ , \tag{3.8}$$

$$\zeta_j = t \exp\left(\frac{2\pi ij}{m}\right) \ , \ 0 \le j \le m-1 \ , \ i = \sqrt{-1} \ . \tag{3.9}$$

By using equation (3.8) and the assumption that all eigenvalues of A are algebraically simple, it is not hard to show that

$$\{w \in \mathbb{C}^n : (I-t^m A^m)^j(w)=0 \ \text{ for some } \ j\} = \operatorname{span}\{w_\zeta : \zeta^m = t^m\} \ , \tag{3.10}$$

where w_ζ is an eigenvector of A if ζ^{-1} is an eigenvalue and otherwise $w_\zeta = 0$. Because m is odd, $\zeta = t$ is the only real solution of $\zeta^m = t^m$; otherwise the values of ζ such that ζ^{-1} is an eigenvalue and $\zeta^m = t^m$ occur in nonreal, conjugate pairs. Because one can also easily show using equation (3.8) that

$$\{w_\zeta : \zeta^m = t^m \ , \ \zeta^{-1} \ \text{ is an eigenvalue of } \ A\}$$

is a linearly independent set, one concludes that

$$n_{t^m}(A^m) \equiv \dim\{w\in\mathbb{C}^n : (I-tA)w=0\} \equiv n_t(A) \pmod 2 ,$$

which is the desired result. □

The next step in proving Theorem 3.1 is to show that it suffices to prove it in the case m is a prime.

LEMMA 3.2. *In order to prove Theorem 3.1 it suffices to prove it in the case* $m = p =$ *a prime.*

PROOF. Let notation and assumptions be as in Theorem 3.1 and suppose the theorem has been proved for the case $m = p$. If $m = p^t$, define $\Sigma_j = \{x\in\bar{H} : f^s(x)=x , s=p^j\}$. Because $f^m = (f^s)^r$ $(s=p^j , r=p^{m-j})$, it is easy to see that $\Sigma_j \subset \Sigma_t \subset H$. If $x \in \Sigma_j$, it follows that

$$f(x) \in f(\Sigma_t) \subset \Sigma_t \subset H$$

(because it is assumed that $f(\Sigma_t) \subset \Sigma_t)$, so

$$f^s(f(x) = f(f^s(x)) = f(x)$$

and $f(x) \in \Sigma_j$. Now define $g = f^{p^{j-1}}$, so $g^p = f^{p^j}$; because $g(\Sigma_j) \subset \Sigma_j$, the mod p theorem (for the case $m=p$) implies

$$i_X(f^{p^j},H) \equiv i_X(f^{p^{j-1}},H) \pmod p , \tag{3.11}$$

and because equation (3.11) is valid for $1 \le j \le t$, one obtains

$$i_X(f^{p^t},H) \equiv i_X(f,H) \pmod p ,$$

which is the desired result. □

The next lemma is a technical result concerning approximation of the function f in Theorem 3.1; the basic step in the lemma is simply a tedious epsilon-delta argument.

LEMMA 3.3. *Let* G *be an open subset of* $\mathbf{R}^n$, $f : G \to \mathbf{R}^n$ *a continuous map and* H *a bounded open subset of* G *such that* f^m *is defined on* H *for some* $m \geq 1$, *so* $K = \bigcup_{j=0}^{m-1} f^j(\bar{H})$ *is a compact subset of* G. *Assume that if* $\Sigma_f = \{x \in \bar{H} : f^m(x) = x\}$, *then* $f(\Sigma_f) \subset \Sigma_f \subset H$. *If* U *is a given open neighborhood of* K *such that* $K \subset U \subset G$, *there exists* $\delta > 0$ *such that if* $g : U \to \mathbf{R}^n$ *is any continuous map such that*

$$\|f(x) - g(x)\| < \delta \quad \text{for all} \quad x \in U, \tag{3.12}$$

then

(1) g^m *is defined on* $\bar{H}$ *and if* $\Sigma_g = \{x \in \bar{H} : g^m(x) = x\}$,

$$g(\Sigma_g) \subset \Sigma_g \subset H, \tag{3.13}$$

and

$$\text{(2)} \quad i_X(g^j, H) = i_X(f^j, H) \quad \textit{for} \quad j = 1 \quad \textit{or} \quad m \quad (X = \mathbf{R}^n). \tag{3.14}$$

PROOF. Given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $g : U \to \mathbf{R}^n$ is any continuous map satisfying equation (3.12), then $g^j(\bar{H}) \subset U$ for $0 \leq j \leq m - 1$ and

$$\|f^j(x) - g^j(x)\| < \varepsilon \quad \text{for all} \quad x \in \bar{H} \quad \text{and} \quad 1 \leq j \leq m. \tag{3.15}$$

The proof of the previous statement is elementary, but tedious, and is left to the reader.

Now select V to be an open neighborhood of Σ_f such that $\overline{V} \subset H$ and such that

$$(3.16) \qquad f(\overline{V}) \subset H .$$

Equation (3.16) is possible because $f(\Sigma_f) \subset \Sigma_f$. Select $\eta > 0$ so that η

$$(3.17) \qquad \|f^j(x)-x\| \geq \eta \quad \text{for} \quad j = 1 \quad \text{or} \quad m \quad \text{and for all} \quad x \in \overline{H} - V ,$$

and select $\varepsilon > 0$ such that $\varepsilon < \eta$ and such that the ε-neighborhood of $f(\overline{V})$ is contained in H. If $\delta > 0$ is chosen so that equation (3.15) holds, equation (3.17) and (3.15) imply that

$$(3.18) \qquad \|x-(1-t)f^j(x)-tg^j(x)\| > 0$$

for $j = 1$ or m, $0 \leq t \leq 1$ and all $x \in \overline{H} - V$. Equation (3.18) and the homotopy property for the index imply equation (3.14). Equation (3.18) also implies that $\Sigma_g \subset V$, and $g(V)$ is contained in the ε-neighborhood of $f(V)$, so

$$g(\Sigma_g) \subset H .$$

Thus if $x \in \Sigma_g$, $g(x) \in H$ and

$$g^m(g(x)) = g(g^m(x)) = g(x) ,$$

that is, $g(x) \in \Sigma_g$. □

By using Lemmas 3.2 and 3.3 one sees (by approximating f closely by a C^∞ function on U) that it suffices to prove Theorem 3.1 in the case $m = p$ and f is C^∞, and these assumptions will henceforth be made. The next lemma shows that it suffices to prove Theorem 3.1 when f has no fixed points in H.

LEMMA 3.4. *In order to prove Theorem 3.1 it suffices to prove it when* f *has no fixed points in* H .

PROOF. Let notation and assumptions be as in Theorem 3.1, and suppose that Theorem 3.1 holds when f has no fixed points in H . As already remarked, one can assume f is C^∞ . By using the Sard theorem and Lemma 3.3 one can, by replacing $f(x)$ by $f(x) - a$, where a is a regular value of $I - f$ and $\|a\|$ is small, assume that $f(x)$ has only finitely many fixed points, say $\{x_j : 1 \le j \le \mu\}$ in $\overline{H}$, and that $A_j = I - df(x_j)$, the Fréchet derivative of $I - f$ at x_j , is nonsingular.

For x near x_j one can write

$$f(x) = x_j + A_j(x-x_j) + R_j(x) , \tag{3.19}$$

where

$$\lim_{\|x-x_j\| \to 0} \frac{\|R_j(x)\|}{\|x-x_j\|} = 0 . \tag{3.20}$$

For $1 \le j \le \mu$, there exists a positive constant c_j such that

$$\|(I-A_j)u\| \ge c_j\|u\| \tag{3.21}$$

for all $u \in \mathbf{R}^n$. Select δ as in Lemma 3.3 and for $1 \le j \le \mu$ select r_j , $0 < r_j \le 1$, such that the open balls $B_{r_j}(x_j)$ of radius r_j and center x_j are pairwise disjoint and such that

$$\|R_j(x)\| \le \varepsilon_j\|x-x_j\| \quad \text{for} \quad \|x-x_j\| \le r_j , \tag{3.22}$$

where

$$\varepsilon_j = \min(\frac{\delta}{2}, \frac{c_j}{2}) . \tag{3.23}$$

For each j, $1 \le j \le \mu$, select δ_j, $0 < \delta_j\|A_j\| < \varepsilon_j$, such that $I - (1+\delta_j)A_j$ and $I - (1+\delta_j)^p A_j^p$ are nonsingular. Finally, select ρ_j, $0 < \rho_j < r_j$ and observe that because $\{B_{r_j}(x_j) : 1 \le j \le \mu\}$ and $G - \bigcup_{j=1}^{\mu} \overline{B_{\rho_j}(x_j)}$ gives a finite open covering of G, there exists a C^∞ partition of unity subordinate to this covering. More precisely, there exist nonnegative C^∞ functions $\varphi_j(x)$ for $1 \le j \le \mu$ and a nonnegative C^∞ function $\varphi(x)$ such that support$(\varphi_j) \subset B_{r_j}(x_j)$, support$(\varphi) \subset G - \bigcup_{j=1}^{\mu} \overline{B_{\beta_j}(x_j)}$ and

$$\sum_{j=1}^{\mu} \varphi_j(x) + \varphi(x) = 1 \quad \text{for all} \quad x \in G .$$

With these technical preliminaries we are almost done. Define $g : G \to \mathbf{R}^n$ by

$$g(x) = \sum_{j=1}^{\mu} \varphi_j(x)[x_j + (1+\delta_j)A_j(x-x_j)] + \varphi(x)f(x) . \tag{3.24}$$

For $x \notin B_{r_j}(x_j)$, $f(x) = g(x)$, while for $x \in B_{r_j}(x_j)$,

$$f(x) = \varphi_j(x)[x_j + A_j(x-x_j) + R_j(x)] + \varphi(x)f(x) . \tag{3.25}$$

It follows that if $x \in B_{r_j}(x_j)$ (so $\varphi_k(x)=0$ for $k \ne j$),

$$\begin{aligned} \|f(x)-g(x)\| &\le \|\delta_j A_j(x-x_j)\| + \|R_j(x)\| \\ &\le (\|\delta_j A_j\| + \varepsilon_j)\|x-x_j\| \\ &\le \delta\|x-x_j\| < \delta , \end{aligned} \tag{3.26}$$

so $\|f(x)-g(x)\| < \delta$ for all $x \in G$. Thus Lemma 3.3 implies

$$i_X(f^j,H) = i_X(g^j,H) \quad \text{for} \quad j = 1 \text{ or } p ,$$

and it suffices to prove Theorem 3.1 for g instead of f.

If $g(x) = x$ for some $x \in \bar{H}$, one must have $x \in B_{r_j}(x_j)$ for some j ($g(x)=f(x)$ outside $\bigcup B_{r_j}(x_j)$. But if $g(x) = x$ for $x \in B_{r_j}(x)$ one has

$$x - x_j - A_j(x-x_j) = \delta_j\varphi_j(x)A_j(x-x_j)+\varphi(x)R_j(x) \ . \tag{3.27}$$

By definition one obtains

$$\|(I-A_j)(x-x_j)\| \geq c_j\|x-x_j\| \quad \text{and} \quad (\text{for } x\neq x_j)$$

$$\|\delta_j\varphi_j(x)A_j(x-x_j)+\varphi(x)R_j(x)\| \leq [\|\delta_jA_j\|+\varepsilon_j]\|x-x_j\| < c_j\|x-x_j\| \ ,$$

so equality can occur in equation (3.27) (for $x \in B_{r_j}(x_j)$) if and only if $x = x_j$. In other words, the fixed points of g in H are the same as the fixed points of f.

Finally, select σ_j, $0 < \sigma_j < \rho_j$ such that

$$g^k(B_{\sigma_j}(x_j)) \subset B_{\rho_j}(x_j) \quad \text{for} \quad 1 \leq k \leq p \ .$$

A simple calculation shows that for $x \in B_{\sigma_j}(x_j)$,

$$g^k(x) = x_j + (1+\delta_j)^k A_j^k(x-x_j) \ , \ 1 \leq k \leq p \ . \tag{3.28}$$

Because of the way δ_j was selected, equation (3.28) implies that x_j is the only fixed point of g^p (and of g) in $\overline{B_{\sigma_j}(x_j)}$, and Lemma 3.1 implies that

$$i_X(g^p, B_{\sigma_j}(x_j)) \equiv i_X(g, B_{\sigma_j}(x_j)) \pmod p \ . \tag{3.29}$$

If one defines $W = H - \bigcup_{j=1}^{\mu} \overline{B_{\sigma_j}(x_j)}$, g has no fixed points in $\overline{W}$, and if one knew that Theorem 3.1 were true for $g|W$, it would follow that

(3.30) $$i_X(g^p,W) \equiv i_X(g,W) \equiv 0 \pmod{p} .$$

Finally, equations (3.29) and (3.30), together with the additivity property for the fixed point index, would give

$$i_X(g^p,H) \equiv i_X(g,H) \pmod{p} ,$$

which is the desired result. □

The previous lemmas show that it suffices to prove Theorem 3.1 when m is a prime, and f is a C^∞ function with no fixed points in $\bar{H}$. The reduction to this point is standard and is used in all proofs of the mod p theorem of which I am aware.

LEMMA 3.5. *Suppose that* f *is as in Theorem 3.1,* p *is a prime, and* f *is a* C^∞ *function with no fixed points in* $\bar{H}$. *Suppose that for every* $\delta > 0$ *there exists a continuous map* $g : H \to X = \mathbf{R}^n$ *such that*

(3.31) $$\sup\{\|f(x)-g(x)\| : x \in H\} < \delta$$

and such that g^p *has only finitely many fixed points in* H. *Then*

$$i_X(f^p,H) \equiv 0 \pmod{p} .$$

PROOF. If $\Sigma_f = \{x \in \bar{H} : f^p(x)=x\}$, it is assumed that $f(\Sigma_f) \subset \Sigma_f \subset H$, so there exists an open neighborhood H_1 of Σ_f such that

(3.32) $$f^j(\bar{H}_1) \subset H \quad \text{for} \quad 0 \le j \le p .$$

The additivity property of the fixed point index implies that

$$i_X(f^p,H) = i_X(f^p,H_1) .$$

If one applies Lemma 3.3 (with H_1 taking the place of H in Lemma 3.3) one find that if δ in equation (3.31) is sufficiently small, then g^p is defined on H_1, $g(\Sigma_g) \subset \Sigma_g \subset H_1$ (where $\Sigma_g = \{x\ \overline{H}_1 : g^p(x)=x\}$),

$$i_X(f^p, H_1) = i_X(g^p, H_1) \tag{3.33}$$

and g has no fixed points in H_1.

If y is a fixed point of g^p in $\overline{H}_1$, then because $g(x) \neq x$ for all x in $\overline{H}_1$ and p is a prime, one can check that the points $x_j = g^j(x)$, $0 \le j \le p-1$ are all distinct and all fixed points of g^p. Because g has only finitely many fixed points, there exist open neighborhoods U_j about x_j, $U_j \subset H_1$, such that x_j is the only fixed point of g^p in U_j. To prove that the right hand side of equation (3.33) is congruent to zero mod p, it suffices (by the additivity property of the fixed point index) to prove

$$i_X(g^p, U_o) = i_X(g^p, U_j)\ ,\quad 1 \le j \le p-1\ . \tag{3.34}$$

For a fixed j, $1 \le j \le p-1$, define

$$h_1 : U_o \to \mathbf{R}^n\ ,\quad h_1(y) = g^j(y)$$

and

$$h_2 : U_j \to \mathbf{R}^n\ ,\quad h_2(y) = g^{p-j}(y)\ .$$

Then one has for $y \in V_o \subset U_o$, V_o an open neighborhood of x,

$$g^p(y) = h_2(h_1(y))\ ,$$

and similarly for $y \in V_j \subset U_j$,

$$g^p(y) = h_1(h_2(y))\ .$$

Thus the commutativity property for the fixed point index implies that

$$i_X(g^p, U_o) = i_X(h_2 h_1, V_o)$$

$$= i_X(h_1 h_2, V_j) = i_X(g^p, U_j) \ ,$$

and the lemma is proved. □

It remains to find approximations g as in Lemma 3.5. At this point I shall follow a suggestion of Tromba [123, p. 488] and use a transversality theorem.

First some definitions are needed: If Y and Z are Banach spaces and $L : Y \to Z$ is a bounded linear map, L is called Fredholm of index zero if $R(L)$, the range of L, is closed, and the codimension of $R(L)$ equals the dimension of the null space of L. If U is an open subset of Y and $f : U \to Z$ is continuously Fréchet differentiable with Fréchet derivative $df(x) : Y \to Z$ at $x \in U$, f is called Fredholm of index zero if $df(x)$ is Fredholm of index zero for all $x \in U$. Because the definition is local, it also makes sense when Y and Z are Banach manifolds.

Now suppose that A is a smooth Banach manifold, W is an open subset of a Banach space, Y is a Banach space and

$$\varphi : A \times W \to Y$$

is a C^1 map. The map φ is called a "Fredholm family of index zero" if for each $a \in A$, the map $x \to \varphi(a,x)$ is a Fredholm map of index zero on W. The map φ is called a zero-proper family if whenever $a_n \to a$ and $\varphi(a_n, x_n) \to 0$, the sequence $\{x_n\}$ has a convergent subsequence which converges to a point in W.

The following theorem has been proved by Tromba [122] when A is a smooth Hilbert manifold. It generalizes earlier results of R. Abraham [1].

THEOREM 3.3. (See [122]). *Suppose that* A *is a smooth Banach manifold,* W *is an open subset of a Banach space* X , Y *is a Banach space and* $\varphi : A \times W \to Y$ *is a zero-proper Fredholm family of index zero. Assume that for each* $z = (a,x) \in \varphi^{-1}(0)$, $d\varphi(z)$ *is onto. Then there exists an open, dense set* $\hat{A} \subset A$ *such that whenever* $a \in \hat{A}$, *all the zeros of* φ_a *are nondegenerate (so that* $\varphi_a(x)=0$ *implies* $d\varphi_a(x)$ *is one-one and onto).*

I shall now use Theorem 3.3 to complete the proof of Theorem 3.1.

PROOF of Theorem 3.1. By the previous lemmas, one can assume that f is C^∞ , $f(x) \neq x$ for all x in $\overline{H}$ and $m = p$. As usual, define Σ_f by

$$\Sigma_f = \{x \in H : f^p(x) = x\} ,$$

and let H_1 be an open neighborhood of Σ_f such that

$$f^j(\overline{H}_1) \subset H \quad \text{for} \quad 0 \le j \le p .$$

Define Z to be the Banach space of C^1 maps $k : \mathbb{R}^n \to \mathbb{R}^n$ which have bounded C^1 norm:

$$\|k\| = \sup_{x \in \mathbb{R}^n} \|k(x)\| + \sup_{x \in \mathbb{R}^n} \|dk(x)\| ,$$

and for $\varepsilon > 0$ define A_ε by

$$A_\varepsilon = \{I+k : k \in Z \text{ and } \|k\| < \varepsilon\} , \tag{3.35}$$

where I is the identity map. The set A_ε is homeomorphic to an open subset of Z and so is a smooth Banach manifold. Given $\gamma \in A_\varepsilon$ and $x \in H$, define

$$\phi(\gamma,x) = x - (\gamma f)^p(x) ,$$

where γf denotes the composition of γ on f . If we can prove that, for ε

sufficiently small, ϕ satisfies the hypotheses of Theorem 3.3, then the proof of Theorem 3.1 will be completed by an application of Lemma 3.5. Take $\varepsilon > 0$ so small that for any $\gamma \in A_\varepsilon$ one has (1) $(\gamma f)(x) \neq x$ for all $x \in \overline{H}$ and (2) if $(\gamma f)^p(x) = x$ for some $x \in \overline{H}$, then $(\gamma f)^j(x) \in H_1$ for $0 \leq j \leq p-1$ and if $x_j = (\gamma f)^j(x)$, then $f(x_j) \neq f(x_k)$ for $0 \leq j < k \leq p-1$. It is easy to see then that φ is a zero-proper Fredholm family of index zero.

To complete the proof, one must prove that if $(\gamma_0, x_0) \in A_\varepsilon \times H$ and

$$x_0 = (\gamma_0 f_0)^p(x_0) ,$$

then the Fréchet derivative of φ at (γ_0, x_0) is onto. Define $x_j = (\gamma_0 f)^j(x_0)$, $0 \leq j \leq p-1$, and take $\Gamma \in Z$ = the tangent space to A_ε and $\xi \in \mathbf{R}^n$. A simple induction proves that

$$\begin{aligned} ((d\varphi)(\gamma_0, x_0))(\Gamma, \xi) &= \xi - (d(\gamma_0 f)^p(x_0))(\xi) \\ &\quad - (\Gamma f)(x_{p-1}) - \sum_{i=1}^{p-1} (d(\gamma_0 f)^i(x_{p-i}))(\Gamma f(x_{p-i-1})) \end{aligned} \tag{3.36}$$

To complete the proof, it suffices to prove that given $\xi \in \mathbf{R}^n$, there exists $\Gamma \in Z$ such that the right hand side of equation (3.36) equals ξ. However, by construction all the points $f(x_j)$ are distinct for $0 \leq j \leq p-1$, so one can take Γ to be a map such that $\Gamma(f(x_j)) = 0$ for $0 \leq j < p-1$ and

$$\Gamma(f(x_{p-1})) = -(d(\gamma_0 f)^p(x_0))(\xi) ,$$

which proves $d\varphi(\gamma_0, x_0)$ is onto. □

REMARK 3.1. The reader may have noticed that Theorem 3.3 is proved in [122] for Hilbert manifolds A, whereas the application here is to a non-Hilbert manifold. Thus it may be worth noting that with more care one can reduce to the case that A is finite dimensional. To see this, take $\eta > 0$ and let

$\{y_1,\ldots,y_s\}$ be an η-net for $\overline{H}$; let $\theta_1,\ldots,\theta_s$ be fixed nonnegative C^∞ functions such that $\mathrm{supp}(\theta_j) \subset B_\eta(y_j)$ and $\sum_j \theta_j(y) = 1$ for all $y \in \overline{H}$. Define Z to be the finite dimensional vector space of functions $\theta(y)$ of the form

$$\theta(y) = \sum_{j=1}^{s} \theta_j(y)\xi_j ,$$

where ξ_j are arbitrary vectors in $\mathbf{R}^n$ and let A_ε be defined by equation (3.35), but for this space Z . If η is chosen small enough, one can prove that a variant of the previous argument still applies, but now one is only using the finite dimensional version of Theorem 3.3.

The term "asymptotic fixed point theory" has sometimes been used to describe theorems in which the existence of a fixed point of a map f is derived from assumptions about iterates of f . As might be expected, the mod p theorem is a useful tool in proving asymptotic fixed point theorems. The following result is a typical, simple application of the mod p theorem in this direction. In the statement of the following corollary, recall that P , the set of periodic points of a map $\theta : X \to X$, is defined as

$$P = \{x \in X : \theta^m(x) = x \text{ for some } m \geq 1\} . \tag{3.37}$$

COROLLARY 3.1. *Let* X *be a metric* ANR, $\theta : X \to X$ *a continuous map and* G *an open subset of* X *such that* $\theta|G$ *is compact. If* P *is the set of periodic points of* θ *(see equation (3.37)), assume that there exists a compact set* A *such that* $(P \cap G) \subset A \subset G$, $\theta(P \cap G) \subset G$, $\theta^m(G) \subset A$ *for all* $m \geq m_0$, *and* A *is homotopic in* G *to a point. Then one has*

$$i_X(\theta, G) = 1 .$$

PROOF. For each prime p, Theorem 3.2 implies that

$$i_X(\theta^p, G) \equiv i_X(\theta, G) \pmod{p},$$

so to complete the proof, it suffices to prove that

$$i_X(\theta^p, G) = 1 \quad \text{for} \quad p \geq m_0 . \tag{3.38}$$

The assumption on A means that there is a continuous map $H : A \times [0,1] \to G$ such that $H(x,0) = x$ for all $x \in A$ and $H(x,1) = x_0$ for some $x_0 \in G$ and all $x \in A$. Because $\theta^m(G) \subset A$ for all $m \geq m_0$, H provides a homotopy $\Gamma(x,t)$ (for $p \geq m_0$) of θ^p to the constant map $x \to x_0$ by

$$H(f^p(x),t) = \Gamma(x,t) , \quad (x,t) \in G \times [0,1] .$$

Equation (3.38) now follows from the homotopy property of the fixed point index. □

Corollary 3.1 is not directly suited for later applications. With these applications in mind it will be useful to make some further definitions and obtain some other corollaries of the mod p theorem. The following definition is a slight modification of a definition of F.E. Browder [22].

DEFINITION 3.1. Suppose that C is a topological space, W is an open subset of C, $x_0 \in W$ and $f : W - \{x_0\} \to C$ is a continuous map. Then x_0 is called an *ejective point of* f if there exists a neighborhood U of x_0 such that for each $x \in U - \{x_0\}$ there exists an integer $m = m(x)$ such that $f^m(x)$ is defined and $f^m(x) \notin U$. If f is, in addition, continuous at x_0 and $f(x_0) = x_0$, then x_0 will be called an *ejective fixed point of* f.

It is observed in [96] (see Sections 1 and 3) that in some applications f may *not* be continuous at x_0, but for simplicity continuity will be assumed here.

An open neighborhood of x_o such as U in Definition 3.1 will be called "an ejective neighborhood of x_o for f". If U is an open neighborhood of x_o and the conditions of Definition 3.1 are satisfied if U in Definition 3.1 is replaced by $\bar{U}$, then $\bar{U}$ will be called a *closed ejective neighborhood of* x_o *for* f.

The next result is a point set topology lemma of F.E. Browder [22].

LEMMA 3.6. (Browder [22]). *Let* C *be a compact Hausdorff space,* $f : C \to C$ *a continuous map and* $x_o \in C$ *an ejective fixed point of* f. *Assume that* $f(x) \neq x_o$ *for all* $x \neq x_o$. *Then there exists an open neighborhood* W *of* x_o *such that for any open neighborhood* V *of* x_o *there exists an integer* $m(V)$ *for which* $f^m(C-V) \subset C - W$ *for all* $m \geq m(V)$.

PROOF. Assume for the moment that if $\bar{U}$ is any closed ejective neighborhood of x_o for f, there exists a compact set $C_1 \subset C$, $x_o \notin C_1$, such that $f(C_1) \subset C_1$ and $C - U \subset C_1$. For a fixed U, take W to be an open neighborhood of x_o disjoint from C_1 and let V be an open neighborhood of x_o. If $y \in C - U$ one has $f^m(y) \in C_1$ for all $m \geq 1$. If $y \in A = (C-V) \cap \bar{U}$, there exists an open neighborhood W_y of y and an integer $m = m(y)$ such that $f^m(z) \in C - \bar{U}$ for all $z \in W_y$. It follows that $f^n(z) \in C_1$ for all $n \geq m(y)$ and $z \in W_y$. A simple compactness argument applied to A shows that there exist $y_1, \ldots, y_p \in A$ with

$$A \subset \bigcup_{j=1}^{p} W_{y_j} .$$

For $n \geq \max\{m(y_j) : 1 \leq j \leq p\} = m(V)$ one has that

$$f^n(A) \subset C_1 ,$$

and therefore

$$f^n(C-V) \subset C_1 \subset C - W \ , \ n \geq m(V) \ .$$

Thus to complete the proof it suffices to prove the existence of C_1. If $\bar{U}$ is a closed, ejective neighborhood of x_o and if $z \in B = f(C-U) \cap \bar{U}$, there is an open neighborhood N_z of z and an integer $\mu_z \geq 1$ such that $f^{\mu_z}(x) \in C - \bar{U}$ for all $x \in N_z$. Because B is compact, there is a finite covering of B by sets N_{z_j}, $1 \leq j \leq k$, with corresponding integers μ_{z_j}, $1 \leq j \leq k$. If $\nu = \max\{\mu_{z_j} : 1 \leq j \leq k\}$, define C_1 by

$$C_1 = \bigcup_{j=0}^{\nu} f^j(C-U) \ . \tag{3.39}$$

Note that C_1 is compact, $C - U \subset C_1$ and $x_o \notin C_1$ (because $f(x) \neq x_o$ for $x \neq x_o$). It only remains to prove that $f(C_1) \subset C_1$ and to prove the latter inclusion it suffices to show that if $z \in f^{\nu}(C-U)$, then $f(z) \in C_1$. Thus suppose that $z = f^{\nu}(y)$ for some $y \in C - U$. If $f(y) \in C - U$, $f(z) \in f^{\nu}(C-U) \subset C_1$. If $f(y) \notin C - U$, then $f(y) \in B$ and there exists an integer μ_{z_j} for some j, $1 \leq j \leq k$, such that (writing $\mu = \mu_{z_j}$) $f^{\mu}(f(y)) = f^{\mu+1}(y) \in C - U$. But then one has

$$f(z) = f^{\nu+1}(y) = f^{\nu-\mu}(f^{\mu+1}(y)) \subset f^{\nu-\mu}(C-U) \subset C_1 \ .$$

It follows that $f(z) \in C_1$ for any $z \in f^{\nu}(C-U)$, and the proof is complete. □

In [22], F.E. Browder has proved that if C is a compact, convex subset of a Banach space, $f : C \to C$ a continuous map and $x_o \in C$ an ejective fixed point of f such that x_o is also an extreme point, then f has a fixed point other then x_o. (Recall that $x_o \in C$ is called an "extreme point of C" if there do not exist y, $z \in C$, $y \neq x_o$ and $z \neq x_o$, such that $x_o = (\frac{1}{2})(y+z)$). If C

is not finite dimensional, Browder's theorem is true without the assumption that x_0 is an extreme point of C, but if C is finite dimensional one can easily construct examples to prove that, in any event, x_0 cannot be an interior point of C if Browder's theorem is to be true.

The next theorem is a special case of results in Section 1 of [96]. The theorem contains Browder's result as a special case, but its importance for us will be in the information it provides about the fixed point index of a map on a neighborhood of an ejective fixed point.

THEOREM 3.4. (See [96]). *Let* K *be a closed, convex subset of a Banach space,* W *a relatively open subset of* K *and* $f : W \to K$ *a compact map. Assume* $x_0 \in K$ *is an ejective fixed point of* f *and that* x_0 *is an extreme point of* K. *If* U *is any relatively open neighborhood of* x_0 *such that* x_0 *is the only fixed point of* f *in* U *(neighborhoods like* U *exist by definition of ejectivity) and if* $K \neq \{x_0\}$, *then* $i_K(f,U) = 0$.

PROOF. The additivity property for the fixed point index implies that $i_K(f,U)$ is independent of U for U as in the statement of the theorem, so it suffices to prove the theorem for a sufficiently small neighborhood U. If $\rho > 0$, define $V_\rho = \{x \in K : \|x-x_0\| \leq \rho\}$ and select $\rho > 0$ such that $V_\rho \subset W$. Select U such that $\overline{U}$ is a closed, ejective neighborhood of x_0 for f, $\overline{U} \subset V_\rho$ and $\overline{f(U)} \subset V_\rho$. There exists a continuous retraction $R : K \to V_\rho$ defined by

$$R(x) = \begin{cases} x & \text{for } \|x-x_0\| \leq \rho \\ tx + (1-t)x_0, \quad t = \dfrac{\rho}{\|x-x_0\|}, & \text{for } \|x-x_0\| > \rho . \end{cases}$$

If one defines $g(x) = R(f(x))$, then one can easily check that $g(x) = f(x)$ for all $x \in \overline{U}$ (so U is a closed, ejective neighborhood of x_o for g) and

$$i_K(g,U) = i_K(f,U) .$$

Also, if one defines $C = \overline{co}\, g(W)$, C is compact and convex and one has

$$g(U) \subset g(W) \subset C \subset V_\rho \subset W , \tag{3.40}$$

so the commutativity property of the index gives

$$i_K(g,U) = i_C(g,U \cap C) .$$

However, equation (3.40) also implies that g is defined on C and $g(C) \subset C$. Of course x_o remains an extreme point of C.

The reduction of the previous paragraph to the case $g : C \to C$ shows that one may as well assume from the start that K is compact and convex, that $f : K \to K$ is continuous and that $x_o \in K$ is an extreme point of K and x_o is also an ejective fixed point of f (with closed ejective neighborhood $\overline{U}$). Define $\rho(x) = d(x,\overline{U})$ (the distance of x to $\overline{U}$), take $x_1 \in K$ such that $x_1 \neq x_o$ and select $\varepsilon > 0$ so small that $\varepsilon\rho(x) < 1$ for all $x \in K$. Define $f_\varepsilon : K \to K$ by

$$f_\varepsilon(x) = (1-\varepsilon\rho(x))f(x) + \varepsilon\rho(x)x_1 .$$

Because f_ε agrees with f on $\overline{U}$ one certainly has

$$i_K(f_\varepsilon,U) = i_K(f,U) ,$$

and $\overline{U}$ is a closed ejective neighborhood for f_ε. Moreover, one has that

$$f_\varepsilon(x) \neq x_o \tag{3.41}$$

for all $x \in K - \{x_0\}$: if $x \notin \overline{U}$, inequality (3.41) is true because x_0 is an extreme point, and if $x \in \overline{U}$, it is true because x_0 is an ejective fixed point of f .

Now one is in the situation of Lemma 3.6, so there exists a relatively open neighborhood U_1 of x_0 , $\overline{U}_1 \subset U$, such that for any relatively open neighborhood V of x_0 , $f_\varepsilon^m(K-V) \subset K - U_1$ for all $m \geq m(V)$. Define a set A by

$$A = \{(1-t)y+tx_1 : 0\leq t\leq 1 , y\in K-U_1\} .$$

Clearly A is a compact subset of K and $x_0 \notin A$ (because x_0 is an extreme point). Select V to be a relatively open neighborhood of x_0 such that $\overline{V}$ and A are disjoint. If, in the notation of Corollary 3.1, one defines $X = K$, a metric ANR, $G = K - \overline{V}$, an open subset of X and $\theta = f_\varepsilon$, one can easily check that the hypotheses of Corollary 3.1 are satisfied, so

$$i_K(f_\varepsilon, K-\overline{V}) = 1 . \tag{3.42}$$

On the other hand, $f_\varepsilon : K \to K$ is homotopic to a point, so

$$i_K(f_\varepsilon, K) = 1 . \tag{3.43}$$

The additivity property implies that

$$i_K(f_\varepsilon,V) + i_K(f_\varepsilon,K-\overline{V}) = i_K(f_\varepsilon,V) ,$$

so one concludes that

$$i_K(f_\varepsilon,V) = 0 . \quad \square$$

REMARK 3.2. It is proved in Section 1 of [96] that if K is infinite dimensional (that is, K is not contained in any finite dimensional affine linear subspace) then Theorem 3.4 is true without the assumption that x_0 be an extreme point of K. Also, it suffices to assume that $f|W$ is a strict-set-contraction w.r.t. a generalized measure of noncompactness. If K is finite dimensional, it suffices to assume that x_0 is an ejective fixed point and x_0 is not in the interior of K (the interior is taken with respect to the affine linear subspace spanned by K).

The applications in Section 5 of these notes will also require the notion of an "attractive fixed point", which is more or less the antithesis of an ejective fixed point.

DEFINITION 3.2. Suppose that C is a topological space, W is an open subset of C such that $x_0 \in W$ and $f : W \to C$ is a continuous map such that $f(x_0) = x_0$. The point x_0 is called an *attractive fixed point of* f if there exists an open neighborhood U of x_0, $U \subset W$, such that (1) $f^m(U) \subset W$ for all $m \geq 1$, and (2) if V is any open neighborhood of x_0 there exists an integer $m(V)$ such that $f^m(U) \subset V$ for all $m \geq m(V)$.

THEOREM 3.5. *Let* C *be a closed, convex subset of a Banach space,* W *a relatively open subset of* C *and* $f : W \to C$ *a compact map. Assume that* $x_0 \in W$ *is an attractive fixed point of* f_0. *If* V *is any relatively open neighborhood of* x_0 *such that* x_0 *is the only fixed point of* f *in* V *(such neighborhoods exist by attractivity), then* $i_C(f,V) = 1$.

PROOF. The additivity property of the fixed point index shows that $i_C(f,V)$ is independent of the particular V chosen as in the statement of the theorem. Let U be as in Definition 3.2 and select $\rho > 0$ so small that $\overline{V} \subset U$, where

$$V = \{x \in C : \|x - x_0\| < \rho\} .$$

By definition of attractivity, there exists m_o such that for all $m \geq m_o$, $\overline{f^m(V)} \subset V$. One can verify that the hypotheses of Theorem 3.2 are satisfied for $p \geq m_o$, p a prime, so

$$i_C(f^p,V) \equiv i_C(f,V) \pmod{p} ,$$

and it suffices to prove that

$$i_C(f^p,V) = 1 .$$

However, one has $f^p(\overline{V}) \subset V$ so the commutativity property gives

$$i_C(f^p,V) = i_{\overline{V}}(f^p,V) = i_{\overline{V}}(f^p,\overline{V}) .$$

Because $f^p : \overline{V} \to \overline{V}$ is a compact map which is homotopic to a point ($\overline{V}$ is convex), the homotopy property implies

$$i_{\overline{V}}(f^p,\overline{V}) = 1 ,$$

and the proof is complete. □

Section 4

GLOBAL BIFURCATION THEOREMS IN METRIC ANR'S

Some time ago P. Rabinowitz [109] proved his now famous "global bifurcation theorem". In Rabinowitz's framework one has a compact map $f : \mathbb{R} \times Y \to Y$, where Y is a Banach space. One assumes that $f(\lambda,0) = 0$ for all real λ and that the map $y \to f(\lambda,y)$ has a Fréchet derivative at $x = 0$ given by $df_y(\lambda,0) = \lambda L$. Rabinowitz's theorem then discusses the structure of connected components of the closure of $\{(\lambda,y)\in\mathbb{R}\times Y) : f(\lambda,y)=y\ ,\ y\neq 0\}$. In some applications the map f is not naturally defined on $\mathbb{R} \times Y$, Y a Banach space, but on a space $J \times X$, where J is an open interval of reals and X is a metric ANR. For example, X may be a closed convex subset of a Banach space. One can assume that f is locally compact and that $f(\lambda,x_o) = x_o$ for all $\lambda \in J$ and some $x_o \in X$, and one can again consider the closure in $J \times X$ of $\{(\lambda,x)\in J\times X : f(\lambda,x)=x \text{ and } x\neq x_o\}$. Even assuming that one can give a meaning to the Fréchet differentiability of the map $x \to f(\lambda,x)$ at $x = x_o$ (as will be the case when X is a cone and $x_o=0$), it turns out that in the application to be discussed in the next section the map $x \to f(\lambda,x)$ is *not* Fréchet differentiable at x_o. Nevertheless, as was first observed in [99], the fixed point index and fixed point theorems can be used to give useful generalizations of Rabinowitz's theorem to metric ANR's. (The statements of results in [99] were for the case that X is a closed convex subset of a Banach space, but, as will be seen here, the arguments were valid more generally). The case that X is a cone in Banach space is important in applications, and Dancer [32] and Turner [124] have (independently of [99]) given generalizations (for cone mappings which are differentiable at 0) of Rabinowitz's global bifurcation theorem.

In order to state the theorems of this section, it is convenient to collect some hypotheses. Suppose that X is a metric ANR with metric d, J is an open interval of reals and $f : J \times X \to X$ is a continuous, locally compact map such that $f(\lambda, x_0) = x_0$ for all $\lambda \in J$ and some $x_0 \in X$. Assume that there exists a countable subset Λ of J such that (i) if J_0 is any compact interval with $J_0 \subset J$, then $J_0 \cap \Lambda$ is a finite set and (ii) if I_0 is any compact interval of reals such that $I_0 \subset J$ and $I_0 \cap \Lambda$ is empty, then there exists $\varepsilon = \varepsilon(I_0) > 0$ such that $f(\lambda, x) \neq x$ for (λ, x) such that $\lambda \in I_0$ and $0 < d(x, x_0) \leq \varepsilon$.

DEFINITION 4.1. If $f : J \times X \to X$ is as in the preceding paragraph, we shall say that f satisfies H1.

If f satisfies H1, one says that bifurcation occurs at (λ_0, x_0) if, for any open neighborhood U of (λ_0, x_0) in $J \times X$ there exists $(\lambda, x) \in U$ such that $f(\lambda, x) = x$ and $x \neq x_0$. Clearly, if bifurcation occurs at (λ_0, x_0), one must have $\lambda_0 \in \Lambda$, but this is not sufficient for bifurcation to occur.

If $x \in X$ and $r > 0$ it will be convenient to write
$B_r(x) = \{y \in X : d(y,x) < r\}$ and $V_r(x) = \{y \in X : d(y,x) \leq r\}$.

LEMMA 4.1. *Assume that* f *satisfies* H1 *and that* J_0 *is a compact interval,* $J_0 \subset J$, *such that* $\Lambda \cap J_0 = \{\lambda_0\}$. *If* $\lambda \in J_0$ *and* $\lambda \neq \lambda_0$, *there exists* $\rho(\lambda) > 0$ *such that* $f_\lambda(x) \equiv f(\lambda, x) \neq x$ *for* $0 < d(x, x_0) \leq \rho(\lambda)$. *Furthermore, writing* $B_{\rho(\lambda)}$ *for* $B_{\rho(\lambda)}(x_0)$, $i_X(f_\lambda, B_{\rho(\lambda)})$ *is constant for* $\lambda > \lambda_0$

and $\lambda \in J_o$, *and* $i_X(f_\lambda, B_{\rho(\lambda)})$ *is constant for* $\lambda < \lambda_o$ *and* $\lambda \in J_o$.

PROOF. The existence of $\rho(\lambda) > 0$ follows from the definition of H1 . If $\lambda_o < \lambda_1 < \lambda_2$, $\lambda_i \in J_o$, H1 implies that there exists $\varepsilon > 0$ such that $f_\lambda(x) \neq x$ for $\lambda_1 \leq \lambda \leq \lambda_2$ and $0 < d(x,x_o) \leq \varepsilon$. The homotopy property of the fixed point index implies that

$$i_X(F_{\lambda_1}, B_\varepsilon) = i_X(f_{\ 2}, B_\varepsilon) ,$$

and the additivity property gives

$$i_X(f_{\lambda_i}, B_\varepsilon) = i_X(f_{\lambda_i}, B_{\rho(\lambda_i)}) , \ i = 1,2 .$$

Thus one obtains that

$$i_X(f_{\lambda_1}, B_{\rho(\lambda_1)}) = i_X(f_{\lambda_2}, B_{\rho(\lambda_2)}) .$$

An analogous proof shows that $i_X(f_\lambda, B_{\rho(\lambda)})$ is constant for $\lambda < \lambda_o$ and $\lambda \in J_o$.□

Suppose now that f satisfies H1 , that $\lambda_o \in \Lambda$ and that $\rho(\lambda)$ and J_o are as in the preceding lemma. Select $\lambda_1 \in J_o$, $\lambda_1 < \lambda_o$ and $\lambda_2 \in J_o$ with $\lambda_2 > \lambda_o$.

DEFINITION 4.2. $\Delta(\lambda_o) = i_X(f_{\lambda_2}, B_{\rho(\lambda_2)}) - i_X(f_{\lambda_1}, B_{\rho(\lambda_1)})$, where $f_\lambda(x) \equiv f(\lambda,x)$ and $B_{\rho(\lambda)} \equiv B_{\rho(\lambda)}(x_o)$.

By using Lemma 4.1 one can see that the definition of $\Delta(\lambda_o)$ is independent of the particular λ_1 and λ_2 chosen as above. Furthermore, $\rho(\lambda_i)$ in the definition can be replaced by any number $r > 0$ such that $f(\lambda_i, x) \neq x$ for $0 < d(x,x_o) \leq r$.

Before proving the analogue of Rabinowitz's theorem for metric ANR's, it is necessary to recall a point set topology result. The following lemma has been attributed by analysts to Whyburn [125, Chapter 1]. But as J.C. Alexander [3] has pointed out, it actually goes back to the early days of point set topology: see chapter five of [78]. In Corollary 4 of [3], Alexander shows that the assumption of metrizability, which is crucial for Whyburn's proof, is actually irrelevant.

LEMMA 4.2. (See [3] and the references there). *Assume that* M *is a compact Hausdorff space and that* A *and* B *are disjoint, closed nonempty subsets of* M . *Then either there exist disjoint, closed subsets* K_A *and* K_B *of* M *such that* $A \subset K_A$, $B \subset K_B$ *and* $M = K_A \cup K_B$ *or there exists a connected subset* D *of* M *such that* $D \cap A$ *and* $D \cap B$ *are nonempty.*

In the above lemma, recall that a topological Hausdorff space D is called "connected" if there do not exist disjoint nonempty open sets U and V such that $D = U \cup V$. In Lemma 4.2, a subset D of M inherits a topology from M which makes it a topological space. Recall, also, that if $y \in M$, the connected component of M containing y is by definition the union of all connected subsets C of M such that $y \in C$; since this union is connected, the connected component containing y is the largest connected subset of M containing y .

One problem the reader should bear in mind is that in applying the previous connectivity results the space M may, *a priori*, be very irregular, so caution is necessary in making some "obvious" point set topological assertions.

THEOREM 4.1. *Assume that* f *satisfies* H1 *and define* S *by*

$$S = (\Lambda \times \{x_0\}) \cup \{(\lambda, x) \in J \times X : x \neq x_0 \text{ and } f(\lambda, x) = x\} . \tag{4.1}$$

Given $\lambda_0 \in \Lambda$ *let* S_0 *be the connected component of* S *which contains* (λ_0, x_0) .

Then either (a) S_o *is not compact or* (b) S_o *is compact and if* $\Lambda_o := \{\lambda \epsilon \Lambda : (\lambda, x_o) \epsilon S_o\}$ *is a finite set and*

(4.2) $$\sum_{\lambda \epsilon \Lambda_o} \Delta(\lambda) = 0 .$$

If S_o *is compact and* $\Delta(\lambda_o) \neq 0$, *then* S_o *contains a point* (λ_1, x_o) *such that* $\lambda_1 \in \Lambda_o$ *and* $\lambda_1 \neq \lambda_o$

Note that in either case one can easily prove that S and S_o are closed.

PROOF of Theorem 4.1. Assume S_o is compact, so it suffices to prove eq. (4.2). If $J = (a,b)$, where $-\infty \le a < b \le +\infty$, there exist finite numbers c and d , $a < c < d < b$, such that if $(\lambda, x) \in S_o$, then $c < \lambda < d$. This follows from compactness of S_o , because $\{\lambda : (\lambda, x) \epsilon S_o\}$ is a compact subset of J (it is the continuous image of a compact map). The assumption H1 implies that $\Lambda \cap [c,d]$ is finite, so Λ_o is certainly a finite set. Because Λ_o is finite, one can write

(4.3) $$\Lambda_o = \{\lambda_i : 1 \le i \le m\}$$

and assume that $\lambda_i < \lambda_{i+1}$ for $1 \le i \le m-1$. Also, there exists $\eta > 0$ such that if $\lambda \in \Lambda_o$, then

(4.4) $$[\lambda - \eta, \lambda + \eta] \cap \Lambda = \{\lambda\} .$$

Extend the metric d to $J \times X$ by defining

$$d((s,x),(t,y)) = |t-s| + d(x,y)$$

and for $\varepsilon > 0$ define U_ε by

(4.5) $$U_\varepsilon = \{(s,x) \epsilon J \times X : d((s,x), S_o) \le \varepsilon\} .$$

By selecting ε small enough one can assume that $f|U_\varepsilon$ is compact, that

$$(4.6) \qquad U_\varepsilon \cap (J\times\{x_0\}) \subset \bigcup_{\lambda\in\Lambda_0} ((\lambda-\eta,\lambda+\eta)\times\{x_0\})$$

and that $c < t < d$ for (t,x) in U_ε .

The crucial step in the proof, as in Rabinowitz's argument, is to find an open neighborhood Ω of S_0 such that $f(s,x) \neq x$ for $(s,x) \in \partial\Omega$ unless $x \neq x_0$ and $|s-\lambda| \leq \eta$ for some $\lambda \in \Lambda_0$. This is "obvious" if S is "regular", but in general, caution is necessary. The idea is to apply Lemma 4.2. In the notation of Lemma 4.2, define $M = S \cap U_\varepsilon$, $A = S_0$ and $B = S \cap \partial U_\varepsilon$, so M is a compact Hausdorff space with closed, disjoint subsets A and B . One can assume that B is nonempty: otherwise take Ω to be the interior of U_ε . The definition of S_0 implies that there does not exist a connected D in M which has nonempty intersection with A and B , so Lemma 4.2 implies that there exist closed, disjoint subsets K_A and K_B of M such that $A \subset K_A$, $B \subset K_B$ and $M = K_A \cup K_B$. Because K_A and K_B are closed and disjoint and $K_A \cap \partial U_\varepsilon$ is empty, there exists an open neighborhood Ω of K_A in $J \times X$ such that $\overline{\Omega} \cap \partial U_\varepsilon$ is empty and $\overline{\Omega} \subset U_\varepsilon$ and $c < t < d$ for $(t,x) \in \overline{\Omega}$. The construction of Ω insures that if $(t,x) \in \partial\Omega$, then $(t,x) \notin K_A \cup K_B$, so $(t,x) \notin S$. In particular, if $f(t,x) = x$ for $x \in \partial\Omega$, one must have $x = x_0$ and $|t-\lambda| < \eta$ for some $\lambda \in \Lambda_0$.

For notational convenience, define $\Omega_t = \{x : (t,x)\in\Omega\}$ and note that Ω_t is empty for $t \leq c$ or $t \geq d$. As usual, one defines $f_t(x) = f(t,x)$. Because Ω is an open neighborhood of (λ,x_0) for $\lambda \in \Lambda_0$, there exist positive numbers $\eta_1 < \eta$ and r such that for all $\lambda \in \Lambda_0$

$$(4.7) \qquad [\lambda-\eta_1,\lambda+\eta_1] \times V_r(x_0) \subset \Omega \ , \ \lambda \in \Lambda_0 \ .$$

Because of hypothesis H1, there exists a positive number $\rho < r$ such that

$$f(t,x) \neq x \tag{4.8}$$

if $\eta_1 \leq |t-\lambda| \leq \eta$ for some $\lambda \in \Lambda_o$ and $0 < d(x,x_o) \leq \rho$. By decreasing ρ one can also assume that f is compact on $[\lambda-\eta,\lambda+\eta] \times V_\rho$.

The idea of the proof now is to apply the homotopy property for the fixed point index on appropriate intervals. First, suppose $\lambda \in \Lambda$ and define an open subset W of $[\lambda-\eta,\lambda+\eta] \times X$ by

$$W = \{(t,x)\in[\lambda-\eta,\lambda+\eta]\times X : (t,x)\in\Omega \text{ or } d(x,x_o)<\rho\} .$$

If $(t,x) \in \partial W$ and $f(t,x) = x$, then $(t,x) \in \partial\Omega$ or $|t-\lambda| \leq \eta$ and $d(x,x_o) = \rho$. If $(t,x) \in \partial\Omega$, $x = 0$ so $(t,x) \notin \partial W$, a contradiction. If $d(x,x_o) = \rho$ and $\eta_1 \leq |t-\lambda| \leq \eta$, the construction insures that $f(t,x) \neq x$. Finally, if $|t-\lambda| \leq \eta_1$ and $d(x,x_o) = \rho$, $(t,x) \notin \partial W$. Thus one has that $f(t,x) \neq x$ for $x \in \partial W$. If one defines $W_t = \{x|(t,x)\in W\}$, the homotopy property implies that

$$i_X(f_{\lambda+\eta},W_{\lambda+\eta}) = i_X(f_{\lambda-\eta},W_{\lambda-\eta}) . \tag{4.9}$$

The additivity property of the fixed point index gives

$$i_X(f_t,W_t) = i_X(t_t,\Omega_t) + i_X(f_t,B_\rho(x_o)) , \quad t = \lambda \pm \eta . \tag{4.10}$$

Combining equations (4.9) and (4.10) and using the definition of $\Delta(\lambda)$ gives

$$i_X(f_{\lambda+\eta},\Omega_{\lambda+\eta}) - i_X(f_{\lambda-\eta},\Omega_{\lambda-\eta}) = -\Delta(\lambda) . \tag{4.11}$$

Next suppose that λ and λ' are consecutive elements of Λ_o so $\Lambda_o \cap (\lambda,\lambda')$ is empty. By our construction we know that $f_t(x) \neq x$ for $(t,x) \in \partial\Omega$

and $\lambda + \eta \le t \le \lambda' - \eta$, so the homotopy property gives

$$(4.12) \qquad i_X(f_{\lambda'-\eta},\Omega_{\lambda'-\eta}) = i_X(f_{\lambda+\eta},\Omega_{\lambda+\eta}) \ .$$

If one applies equations (4.11) and (4.12) repeatedly and recalls eq. (4.3) one obtains for $1 \le j < k \le m$ that

$$(4.13) \qquad i_X(f_{\lambda_k+\eta},\Omega_{\lambda_k+\eta}) - i_X(f_{\lambda_j-\eta},\Omega_{\lambda_j-\eta}) = -\sum_{i=j}^{k} \Delta(\lambda_i) \ .$$

Choosing $j = 1$ and $k = m$ in eq. (4.13) gives

$$(4.14) \qquad i_X(f_{\lambda_m+\eta},\Omega_{\lambda_m+\eta}) - i_X(f_{\lambda_1-\eta},\Omega_{\lambda_1-\eta}) = -\sum_{\lambda\in\Lambda_o} \Delta(\lambda) \ .$$

The homotopy property implies that $i_X(f_t,\Omega_t)$ is constant for $t \ge \lambda_m + \eta$ and for $t \le \lambda_1 - \eta$. Because Ω_t is empty for $t \ge d$ and for $t \le c$, one concludes that

$$(4.15) \qquad i_X(f_t,\Omega_t) = 0 \quad \text{for} \quad t = \lambda_m + \eta \quad \text{or for} \quad t = \lambda_1 - \eta$$

and using this information in eq. (4.14) gives

$$\sum_{\lambda\in\Lambda_o} \Delta(\lambda) = 0 \ ,$$

which is the desired result. □

REMARK 4.1. Theorem 4.1 is essentially Theorem 1.2 in [99], although the theorem in [99] is stated for the case that X is a closed, convex subset of a Banach space.

REMARK 4.2. It may happen that, in the notation of Theorem 4.1, S_o is noncompact but that for some compact interval $[c,d] \subset J$, $\{(s,x)\in S_o : c\le s\le d\}$ is compact. It then follows easily that $\Lambda_o = \{s : c\le s\le d$ and $(s,x_o)\in S_o\}$ is finite. Assume that $c \notin \Lambda_o$ and $d \notin \Lambda_o$. If $\Lambda_o = \{\lambda_i : 1\le i\le m\}$ and $\lambda_i < \lambda_{i+1}$ for

$1 \le i \le m-1$, there exists $\eta > 0$ such that $[\lambda-\eta,\lambda+\eta] \cap \Lambda = \{\lambda\}$ for $\lambda \in \Lambda_o$. If one writes $\Sigma_o = \{(t,x)\in S_o : c\le t\le d\}$ and works in the topological space $[c,d] \times X$, the same proof as in Theorem 4.1 shows that there exists an open neighborhood Ω of Σ_o (in $[c,d]\times X$) such that if $f(t,x) = x$ for some $(t,x) \in \partial\Omega$, then $x = 0$ and $|t-\lambda| < \eta$ for some $\lambda \in \Lambda_o$. The same argument as in Theorem 4.1 now shows that for $1 \le j < k \le m$

$$(4.16) \qquad i_X(f_{\lambda_k+\eta},\Omega_{\lambda_k+\eta}) - i_X(f_{\lambda_j-\eta},\Omega_{\lambda_j-\eta}) = -\sum_{i=j}^{k} \Delta(\lambda_i) .$$

REMARK 4.3. Rabinowitz considered the case that X is a Banach space, $J = \mathbb{R}$ and $f : \mathbb{R} \times X \to X$ is a continuous map which takes bounded sets to sets with compact closure (a compact map). He assumes that $f(t,0) = 0$ for all t and that for t in any compact interval J_o ,

$$f(t,x) = tL(x) + R(t,x) ,$$

where L is a compact linear map and

$$\lim_{\|x\|\to 0} \frac{\|R(t,x)\|}{\|x\|} = 0$$

uniformly for $t \in J$. If $\Lambda = \{t\in\mathbb{R} : I-tL$ is not 1-1$\}$, one can easily verify that Λ satisfies assumption H1. If $t \notin \Lambda$, $\rho(t) = \rho$ is a positive number such that $f(t,x) \ne x$ for $0 < \|x\| \le \rho$, one can prove that

$$(4.17) \qquad \deg(I-f_t,B_\rho,0) = i_X(f_t,B_\rho) = \deg(I-tL,B_\rho,0) ,$$

where $B_\rho = \{x \mid \|x\|<\rho\}$. A classical formula of degree theory (see eq. (3.2) in Section 3, which applies also to compact linear maps) implies that

$$(4.18) \qquad \deg(I-tL,B_\rho,0) = \varepsilon = \pm 1 .$$

Furthermore, if $\lambda_o \in \Lambda$ and $[\lambda_o-\eta,\lambda_o+\eta] \cap \Lambda = \{\lambda_o\}$, one has

$$\deg(I-(\lambda_o+\eta)L,B_\rho,0) = \delta \deg(I-(\lambda_o-\eta)L,B_\rho,0) ,$$

where $\delta = -1$ if λ_o^{-1} is an eigenvalue of odd algebraic multiplicity and $\delta = 1$ if λ_o^{-1} has even algebraic multiplicity. It follows that $\Delta(\lambda_o) = 0$ if λ_o^{-1} has even algebraic multiplicity and $\Delta(\lambda_o) = \pm 2$ if λ_o^{-1} has odd algebraic multiplicity.

Of course there is a generalization of Theorem 4.1 to the case that f is a local strict-set-contraction. Suppose that J is an open interval of real numbers, X is a closed subset of a Banach space Y and $f : J \times X \to X$ is a continuous map. If β is a generalized measure of noncompactness on Y, recall that f is called a local strict-set-contraction (with respect to β) if for every $(t_o,u_o) \in J \times X$ there exist $\varepsilon > 0$ and $r > 0$ such that for any $A \subset V_r(u_o) = \{x \in X : \|x-u_o\| \le r\}$,

$$\beta(f([t_o-\varepsilon,t_o+\varepsilon]\times A)) \le k\beta(A) ,$$

where $k = k(t_o,u_o)$ is a constant less than 1 and possibly dependent on (t_o,u_o).

DEFINITION 4.2. Suppose that X is a closed subset of a Banach space Y and that $X \in \mathcal{F}$ (see Definition 1.1). Assume that J is an open interval of reals, that $f : J \times X \to X$ is a continuous map which is a local strict-set-contraction with respect to a generalized measure of noncompactness β on Y, and that, for some $x_o \in X$, $f(s,x_o) = x_o$ for all $s \in J$. Finally, suppose that $\Lambda \subset J$ is a countable set which satisfies the same assumptions as in hypothesis H1. Then we shall say f satisfies H2.

Essentially the same argument as in Theorem 4.1, with the properties of the fixed point index for local strict-set-contractions substituting for the corresponding properties for locally compact maps, yields:

THEOREM 4.2. *Assume that* $X \in F$ *and* $f : J \times X \to X$ *satisfies* H2. *If* $S = (\Lambda \times \{x_0\}) \cup \{(s,x) \in J \times X : f(s,x) = x \text{ and } x \neq x_0\}$ *and, for some* $\lambda_0 \in \Lambda$, S_0 *is the connected component of* S *which contains* (λ_0, x_0), *then either (a)* S_0 *is noncompact or (b)* S_0 *is compact,* $\Lambda_0 = \{\lambda : (\lambda, x_0) \in S_0\}$ *is finite, and*

$$\sum_{\lambda \in \Lambda_0} \Delta(\lambda) = 0 .$$

In particular, if S_0 *is compact and* $\Delta(\lambda_0) \neq 0$, S_0 *contains a point* (λ_1, x_0) *with* $\lambda_1 \neq \lambda_0$.

Details of the proof of Theorem 4.2 are left to the reader.

In the next section, the problem of finding nontrivial periodic solutions of a parametrized class of nonlinear differential-delay equations will be studied. The following special case of Theorem 4.1 was originally proved in [99] and will be adequate for the applications to differential-delay equations.

COROLLARY 4.1. *Suppose that* C *is a closed, convex subset of a Banach space* Y, $J = (0,\infty)$ *and* $f : J \times C \to C$ *is a continuous map which takes bounded subsets of* $J \times C$ *to precompact sets. Assume that there exists an extreme point* x_0 *of* C *such that* $f(s,x_0) \equiv f_s(x_0) = x_0$ *for all* $s > 0$. *Suppose that for some* $\lambda_0 > 0$, x_0 *is an attractive fixed point of* f_s *for* $0 < s < \lambda_0$ *and an ejective fixed point of* f_s *for* $s > \lambda_0$ *or vice versa. Assume that if* $J_0 \subset (0,\infty)$ *is any compact interval such that* $\lambda_0 \notin J_0$, *then there exists* $\varepsilon = \varepsilon(J_0) > 0$ *such that* $f(s,x) \neq x$ *for* (s,x) *such that* $s \in J_0$ *and* $0 < \|x - x_0\| \leq \varepsilon$. *Finally suppose that if* $(s_k, x_k) \in J \times C$ *is a sequence such that* $s_k \to 0$ *and* $x_k \neq x_0$ *and* $f(x_k, x_k) = x_k$, *then* $\lim_{k\to\infty} \|x_k\| = \infty$. *If* $S = \{(\lambda_0, x_0)\} \cup \{(s,x) \in J \times C : f(s,x) = x \text{ and } x \neq x_0\}$ *and* S_0 *is the connected component of* X *which contains* (λ_0, x_0) *then* S_0 *is unbounded.*

PROOF. It is an easy exercise to see that S_o and S are closed. If x_o is an ejective fixed point of f_s, Theorem 3.4 implies that $i_C(f_s,V) = 0$ for any relatively open neighborhood V of x_o such that $f_s(x) \neq x$ for $x \in V$, $x \neq x_o$. If x_o is an attractive fixed point of f_s, and V is as above, $i_C(f_s,V) = 1$. If x_o is an attractive fixed point of f_s for $0 < s < \lambda_o$ and an ejective fixed point for $s > \lambda_o$, it follows that, in the notation of Theorem 4.1, $\Delta(\lambda_o) = -1$. Similarly, if x_o is an ejective fixed point of f_s for $0 < s < \lambda_o$ and an attractive one for $s > \lambda_o$, $\Delta(\lambda_o) = 1$. Because $\Lambda = \{\lambda_o\}$, Theorem 4.1 implies that S_o is not compact.

It remains to prove S_o is unbounded. If S_o were bounded, the assumptions on f would imply that there exist finite, positive numbers c and d and a number R such that $c \leq t \leq d$ and $\|x\| \leq R$ for all $(t,x) \in S_o$. Because f is compact and

$$\{x | (t,x) \in S_o \text{ for some } t>0\} = f(S_o) ,$$

$S_o \subset [c,d] \times \overline{f(S_o)}$, which is a compact set. Since S_o is a closed subset of a compact set, S_o is compact. This is a contradiction, so S_o must be unbounded. □

If one assumes, as will be the case in the application in Section 5, that for any $M > 0$, $\{(t,x) \in S : 0<t\leq M\}$ is bounded, it follows that for every $\lambda > \lambda_o$ the equation $f(\lambda,x) = x$ has a solution (λ,x) with $x \neq x_o$ and, in fact, (λ,x) can be chosen to lie in S_o. If this were not the case for some $\lambda > \lambda_o$, then connectedness would imply that $S_o \subset \{(t,x) \in J\times C : c<t<\lambda\}$, which would contradict the unboundedness of S_o.

As a next application of Theorem 4.1, suppose that C is a cone with vertex at 0 in a Banach space Y and that $f : (0,\infty) \times C \to C$ is a compact, continuous map (in the sense that bounded sets go to precompact sets under f).

Assume that $f(t,0) = 0$ for all $t > 0$ and that there exists a bounded linear map L such that $L(C) \subset C$ and

$$f(t,x) = tLx + R(t,x) \tag{4.19}$$

where $\lim_{\|x\|\to 0, x\in C} \frac{\|R(t,x)\|}{\|x\|} = 0$ uniformly for t in any compact interval. Define Define Λ by

$$\Lambda = \{t>0 : \exists x\in C-\{0\} \text{ such that } tL(x)=x\} \tag{4.20}$$

and *assume* that Λ is countable and that for any compact interval $J_o \subset (0,\infty)$ $J_o \cap \Lambda$ is finite.

DEFINITION 4.3. If f satisfies the hypothesis of the preceding paragraph, we shall say that f satisfies hypothesis H3.

Some remarks about H3 are in order. It is not hard to prove that $L|C$ is compact, and the proof will be left to the reader. If L is compact as a map of Y to Y, then the various assumptions on Λ in H3 are automatically satisfied. However, as Bonsall's example shows (see Remark 2.1 near the end of Section 2), $L : Y \to Y$ need not be compact even if C is total. In fact, if L and K_γ $(\gamma>0)$ are as in Remark 2.1 and $x_\beta(t) = t^\beta$ for $\beta \geq \gamma$, then $x_\beta \in K_\gamma$ and $Lx_\beta = (\frac{1}{2})^\beta x_\beta$. It follows that in this example

$$\Lambda = \{t>0 : tL(x)=x \text{ for some } x\in K_\gamma-\{0\}\} = [2^\gamma,\infty) ,$$

even though $L|K_\gamma$ is compact and K_γ is total. Thus, in the generality considered here, an explicit assumption on Λ is necessary.

LEMMA 4.3. *Assume that* f *satisfies* H3 *and that* $J_o \subset (0,\infty)$ *is a compact interval such that* $J_o \cap \Lambda$ *is empty. Then there exists* $\varepsilon = \varepsilon(J_o) > 0$ *such that* $f(t,x) \equiv f_t(x) \neq x$ *for* $t \in J_o$ *and* $0 < \|x\| \leq \varepsilon$.

PROOF. If not, there exists a sequence (t_n, x_n) such that $t_n \in J_o$, x_n is nonzero, $\|x_n\| \to 0$ and $f(t_n, x_n) = x_n$. By taking a subsequence, one can assume $t_n \to t$. If one writes $u_n = \frac{x_n}{\|x_n\|}$, eq. (4.19) gives

$$u_n = t_n L u_n + \frac{R(t_n, x_n)}{\|x_n\|} . \tag{4.21}$$

As has been already remarked $L|C$ is compact, so by taking a further subsequence one can assume that $t_n L(u_n)$ converges to $v \in C$ and hence eq. (4.21) implies $u_n \to v$. This gives

$$v = tLv ,$$

where $v \in C$ and $\|v\| = 1$ and contradicts the assumption that $J_o \cap \Lambda$ is empty.□

Lemma 4.3 shows that if f satisfies hypothesis H3, it satisfies H1.

One advantage of doing bifurcation theory in cones or closed, convex sets is that it frequently happens that Λ is a single point or a finite set, whereas in the full Banach space Λ is often the set of inverses of elements of the spectrum of a linear operator. The following proposition, which is a slight variant of results of Krasnosel'skii [74] about "u_o-positive linear operators", illustrates this point.

PROPOSITION 4.1. (See [74]). *Suppose that* C *is a cone in a Banach space* Y *and that* $L : Y \to Y$ *is a bounded linear operator such that* $L(C) \subset C$. *Assume that* $u_1, u_2, \ldots, u_p \in C-\{0\}$ *are such that for any* $x \in C-\{0\}$ *there exists* i, $1 \le i \le p$, *an integer* $m \ge 1$ *and positive reals* α *and* β *such that*

$$\alpha u_i \le L^m x \le \beta u_i . \tag{4.22}$$

If $\Lambda = \{t>0 | tL(x)=x$ for some $x \in C-\{0\}\}$ *, the cardinality of* Λ *is less than or equal to* p *. If* $p = 1$ *, any two eigenvectors of* L *which are elements of* C *are linearly dependent.*

If C is reproducing, $p = 1$ and there exists $x \in C-\{0\}$ such that $Lx = rx$, Krasnosel'skii [74] also proves that the algebraic multiplicity of the eigenvalue r is one.

PROOF of Proposition 4.1. Suppose that $Lx = \lambda x$ for $x \in C-\{0\}$. Equation (4.22) implies that for some positive reals α and β , an integer m and some i , $1 \le i \le p$, one has

(4.31) $$\alpha u_i \le L^m x = \lambda^m x \le \beta u_i ,$$

so $\lambda > 0$. If $Ly = \mu y$ for $y \in C-\{0\}$ and if there exist positive reals α_1 and β_1 and an integer n such that for the same index i as in eq. (4.31) one has

(4.32) $$\alpha_1 u_i \le L^n y = \mu^n y \le \beta_1 u_i ,$$

then to prove $\text{card}(\Lambda) \le p$, it suffices to prove $\lambda = \mu$. Equations (4.31) and (4.32) imply that there exist positive constants c and d such that

(4.33) $$cx \le y \le dx .$$

Suppose, by way of contradiction, that $\mu \ne \lambda$, and for definiteness, assume that $\mu < \lambda$. Applying L^n to both sides of eq. (4.33) gives

(4.34) $$c\lambda^n x \le \mu^n y .$$

Dividing both sides of eq. (4.34) by $c\lambda^n$ and taking the limit as $n \to \infty$, one obtains the contradiction that $-x \in C$.

If $p = 1$, one must prove that y is a multiple of x . Define $s = \sup\{t\geq 0 : tx\leq y\}$; equation (4.33) implies that $s \geq c > 0$. If $y \neq sx$, one has $y - sx \in C-\{0\}$ and by assumption there exists an integer N and positive reals γ and δ such that

$$\gamma u_1 \leq L^N(y-sx) = \lambda^N(y-sx) \leq \delta u_1 ,$$

or

$$y \geq sx + \left(\frac{\gamma}{\lambda^N}\right)u_1 . \tag{4.34}$$

Because eq. (4.31) implies

$$u_1 \geq \left(\frac{\beta}{\lambda^m}\right)x , \tag{4.35}$$

one finds that there exists $s_1 > s$ such that

$$y \geq s_1 x ,$$

contradicting the maximality of s . Thus the original assumption that $y \neq sx$ was wrong. □

The next corollary of Theorem 4.1 is a slight generalization of Theorem 1.3 in [99]. Theorem 1.3 in [99] provides a generalization and, more importantly, a different proof of a result which was proved by Turner [124] by somewhat involved transversality arguments.

COROLLARY 4.2. *Assume that* $f : J \times C \to C$ *satisfies* H3 *and that* $r_C(L) = r > 0$, *where* $r_C(L)$ *denotes the cone spectral radius of* L *and* L *is as in* H3. *If* $f(t_k,x_k) = x_k$ *for a sequence* $(t_k,x_k) \in J \times C$ *such that* $t_k \to 0$ *and* $\|x_k\| > 0$, *assume that* $\lim_{k\to\infty}\|x_k\| = \infty$. *If* $\lambda_0 = r^{-1}$, *then* $\lambda_0 \in \Lambda$; *and if*

$$S = (\Lambda\times\{0\}) \cup \{(t,x)\in J\times C : f(t,x)=x \text{ and } x\neq 0\}$$

and S_o *denotes the connected component of* S *such that* $(\lambda_o, 0) \in S_o$ *, then* S_o *is unbounded.*

PROOF. Corollary 2.4 (Bonsall's theorem) implies that there exists $u \in C-\{0\}$ such that $Lu = ru$, so $\lambda_o \in \Lambda$.

If $t > 0$ and $t \notin \Lambda$, one can easily prove that there exists a positive number a (depending on t) such that

$$\|x-tL(x)\| \geq a\|x\| \ , \ x \in C \ .$$

Using the previous equation and eq. (4.19) one can see that there exists $\varepsilon > 0$ such that

$$(1-s)(tL)(x) + sf_t(x) \neq x$$

for $0 \leq s \leq 1$ and $x \in C$ such that $0 < \|x\| \leq \varepsilon$. Thus the homotopy property implies that

$$i_C(tL, B_\varepsilon) = i_C(f_t, B_\varepsilon) \ , \tag{4.36}$$

where $B_\varepsilon = \{x \in C : \|x\| < \varepsilon\}$. If $t > \lambda_o$ one has

$$\|(tL)^n u\| = (\frac{t}{\lambda_o})^n \|u\| \to \infty$$

as $n \to \infty$, so Theorem 2.2 implies that

$$i_C(tL, B_\varepsilon) = 0 \quad \text{for} \quad t > \lambda_o \ , \ t \notin \Lambda \ . \tag{4.37}$$

Equation (4.37) implies that $\Delta(\lambda) = 0$ for $\lambda \in \Lambda$ and $\lambda > \lambda_o$.

If $0 < t < \lambda_o$, it follows from the definition of $r_C(L)$ that $t \notin \Lambda$ and that $stL(x) \neq x$ for $0 \leq s \leq 1$. Thus the homotopy property implies that

$$i_C(tL, B_\varepsilon) = 1 \ , \ 0 < t < \lambda_0 \ . \tag{4.38}$$

Equations (4.36) - (4.38) yield that $\Delta(\lambda_0) = -1$.

If one applies Theorem 4.1 one sees that S_0 is noncompact, because if S_0 were compact one would have

$$\sum_{\lambda \in \Lambda_0} \Delta(\lambda) = \Delta(\lambda_0) = -1 \ ,$$

contrary to eq. (4.2). The fact that S_0 is unbounded now follows by the same argument as in Corollary 4.1. □

Despite its apparent generality, there are difficulties in applying Theorem 4.1 in some situations. In particular the assumptions on Λ in H1 are sometimes too strong. In order to illustrate this point, I would like to discuss briefly a problem from mathematical biology which is treated in [29, 103, 104, 105] and then, following [104], indicate a way to avoid some of these problems.

Define Y to be the Banach space of continuous, maps of $\mathbb{R}$ to $\mathbb{R}$ which are periodic of fixed period ω , so $y(s+\omega) = y(s)$ for all s . If $y \in Y$, define $\|y\| = \sup_s |y(s)|$. Let C be the cone of nonnegative functions in Y and suppose that $g : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that $g(s,0) = 0$ for all s , $g(s+\omega,x) = g(s,x)$ for all $(s,x) \in \mathbb{R} \times \mathbb{R}^+$ and

$$\lim_{x \to 0^+} \frac{g(s,x)}{x} = a(s) \tag{4.39}$$

uniformly in $s \in \mathbb{R}$, where $a(s)$ in eq. (4.39) is assumed strictly positive and is necessarily periodic of period ω . Assume that $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ and $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ are real analytic functions such that $\alpha(\lambda) < \beta(\lambda)$ for all $\lambda > 0$, $\lim_{\lambda \to 0^+} (\beta(\lambda) - \alpha(\lambda)) = 0$ and $\lim_{\lambda \to \infty} (\beta(\lambda) - \alpha(\lambda)) = \infty$. Let $J = (0,\infty)$ and define

$f : J \times C \to C$ by

$$(4.40) \qquad (f(\lambda,x))(t) = \int_{t-\beta(\lambda)}^{t-\alpha(\lambda)} g(s,x(s))ds .$$

One can easily verify that f is compact and $f(\lambda,0) = 0$. If one defines $L_\lambda : Y \to Y$ by

$$(4.41) \qquad (L_\lambda y)(t) = \int_{t-\beta(\lambda)}^{t-\alpha(\lambda)} a(s)x(s)ds ,$$

it is not hard to prove, using Proposition 4.1, that $(L_\lambda y) = ry$ for some $y \in C-\{0\}$ if and only if $r = r(L_\lambda)$ = the spectral radius of L_λ. From this one concludes that the only possible choice for Λ (in the notation of Theorem 4.1) is $\Lambda = \{\lambda>0 : r(L_\lambda)=1\}$. It is proved in [105] that $\lambda \to r(L_\lambda)$ is C^∞ if $a(s)$ is C^∞, but it is not known (even for relatively simple examples like $\alpha(\lambda) = \frac{\lambda}{2}$, $\beta(\lambda) = \lambda$ and $a(s)=1+\varepsilon \sin s, -1<\varepsilon<1$) whether $\lambda \to r(L_\lambda)$ is real analytic if $a(s)$ is real analytic. Thus, while one can prove Λ is compact, it is not clear that it satisfies the assumptions in hypothesis H1. In fact, an interesting and apparently nontrivial question is what more can be said about Λ ?

In order to circumvent these problems, I would like to weaken the assumptions on Λ in H1. Assume that J is an open interval, X is a metric ANR with metric d and $f : J \times X \to X$ is a locally compact map such that $f(s,x_0) = x_0$ for all $s \in J$ and some $x_0 \in X$. Suppose that $A = \{I_k : 1 \le k < \infty\}$ is a countable, possibly finite, collection of pairwise disjoint, compact intervals $I_k \subset J$ and write $\Lambda = \bigcup_{k=1}^{\infty} I_k$. One allows I_k to be a point. If J_0 is any compact interval such that $J_0 \subset J$ assume that $J_0 \cap I_k$ is empty except for finitely many indices k. If J_0 is a compact interval, $J_0 \subset J$, such that $J_0 \cap \Lambda$ is empty, assume that there exists $\varepsilon = \varepsilon(J_0) > 0$ such that $f(s,x) \ne x$ for $s \in J_0$ and for $0 < d(x,x_0) \le \varepsilon$.

DEFINITION 4.4. If f satisfies the conditions of the preceding paragraph for some collection of intervals A, we shall say that f satisfies H4.

If A is as above and $I_j \in A$, one can define $\Delta(I_j)$. If $I_j = [c_j, d_j]$, select $\eta > 0$ such that $[c_j - \eta, d_j + \eta]$ has empty intersection with I_k for $k \neq j$ and take $\varepsilon > 0$ such that $f_t(x) \neq x$ for $0 < d(x, x_0) \leq \varepsilon$ and for $t = c_j - \eta \equiv \gamma$ or for $t = d_j + \eta \equiv \delta$. If one defines $\Delta(I_j)$ by

$$\Delta(I_j) = i_X(f_\delta, B_\varepsilon(x_0)) - i_X(f_\gamma, B_\varepsilon(x_0)) ,$$

it is easy to see that $\Delta(I_j)$ is independent of the particular ε and η chosen above.

Essentially the same proof as for Theorem 4.1 also gives the following generalization of Theorem 4.1:

THEOREM 4.3. *Assume that* $f : J \times X \to X$ *satisfies* H4 *and that* $A = \{I_k : k \geq 1\}$ *and* Λ *are as in the statement of* H4. *Define* $S = (\Lambda \times \{x_0\}) \cup \{(s,x) \in J \times X : f(s,x) = x \text{ and } x \neq x_0\}$ *and, for a fixed interval* $I_0 \in A$, *define* S_0 *to be the connected component of* S *which contains* $I_0 \times \{x_0\}$. *Then either* (a) S_0 *is noncompact or* (b) S_0 *is compact,* $A_0 = \{I \in A : I \times \{x_0\} \subset S_0\}$ *is finite and*

$$\sum_{I \in A_0} \Delta(I) = 0 .$$

Details of the proof of Theorem 4.3 are left to the reader; a less general but closely related result is proved in [104]. Of course the theorem is also true if f is a local strict-set-contraction and $X \in F$.

If one returns to the motivating example from mathematical biology (equation (4.40)) and selects $c > 0$ and $d > 0$ such that

$$\{\lambda>0 : r(L_\lambda)=1\} \subset [c,d] = I_1 ,$$

then Theorem 4.3 can be applied with $A = \{I_1\}$. One finds that $\Delta(I_1) = -1$, and if f is assumed bounded one finds that for every $\lambda > d$ eq. (4.40) has a nonzero, nonnegative periodic solution of period ω . However, one obtains no information about nontrivial solutions for $c \le \lambda \le d$.

Section 5

PERIODIC SOLUTIONS OF A NONLINEAR DIFFERENTIAL-DELAY EQUATION

In this section I would like to sketch an application of the previous ideas to the problem of finding nontrivial periodic solutions of a differential-delay equation. There are, of course, many other applications, e.g., to partial differential equations [4,35] and integral equations [9, 104,105] but because of considerations of length, I have restricted myself here to one example.

Consider, then, the nonlinear differential-delay equation

$$(5.1)_\lambda \qquad \dot{x}(t) = -\lambda g(x(t))-\lambda f(x(t-1)) \ , \ \lambda > 0 \ .$$

Equations of this type arise in a variety of applications, for example, in mathematical biology [10, 57, 83, 84] and in optics [37, 49, 63, 65, 66, 67]. If $g(x) = x$, eq. $(5.1)_\lambda$ has been the subject of a long paper [86] by John Mallet-Paret and the author and also of an article by Hadeler and Tomiuk [58]. A summary of [86] appears in [87]. Here I shall generalize some of the easier results in [86].

Assume that $f(0) = g(0) = 0$, so that $x = 0$ is always a solution of eq. $(5.11)_\lambda$. I am interested in finding conditions on λ , f and g which insure the existence of a nonconstant periodic solution of eq. $(5.1)_\lambda$. It is proved in

[86, 87] that eq. $(5.1)_\lambda$ may have many periodic solutions for a given λ, so it is useful to specify more precisely the shape of the periodic solution one seeks. Call a periodic solution $x(t)$ of eq. $(5.1)_\lambda$ a "slowly oscillating periodic solution" if $x(0) = 0$ and there exist numbers $z_1 > 1$ and $z_2 > z_1 + 1$ such that $x(t) > 0$ for $0 < t < z_1$, $x(t) < 0$ for $z_1 < t < z_2$ and $x(t+z_2) = x(t)$ for all t. Note that the word "slowly" refers to the assumption that the separation of the zeros of $x(t)$ is greater than one, the time lag. Note also that it is assumed that z_2, the second positive zero of x, is its period. Numerical studies suggest that for some f, g and λ, there also exist periodic solutions $x(t)$ of eq. (5.1) such that $x(0) = 0$ and such that $x(t)$ has positive zeros z_j with $z_{j+1} - z_j > 1$ but for which $x(t)$ has minimal period z_{2m} for some $m > 1$. Very little progress has been made in rigorously proving the existence of such solutions. On the other hand, for reasons which are still incompletely understood, slowly oscillating periodic solutions often seem (numerically) to have striking local or global stability properties; some rigorous results in this direction have been given in [70, 71, 101]. Thus it seems worthwhile to investigate existence of slowly oscillating periodic solutions and some of their properties.

Assume that f and g are differentiable at 0, that $g'(0) = b \geq 0$ and that $f'(0) = c > 0$. If, purely formally, one linearizes eq. $(5.1)_\lambda$ about $x = 0$ one is led to

$$\dot{u}(t) = -\lambda b u(t) - \lambda c u(t-1) \; ; \tag{5.2$_\lambda$}$$

and if one seeks a solution of eq. $(5.2)_\lambda$ of the form $x(t) = e^{zt}$, z complex, one is led to the so-called characteristic equation for eq. $(5.2)_\lambda$:

$$z + \lambda b + \lambda c e^{-z} = 0 \, . \tag{5.3$_\lambda$}$$

It will be necessary to know something about the location of roots of eq. $(5.3)_\lambda$. The needed information can be found in [58] or [86] and, in fact, is probably a "classical" result: see [126], for example, for the case $b = 0$. I shall give a proof only for completeness. The following version of Rouché's theorem will be useful.

LEMMA 5.1. (Rouché's theorem). *Assume that* G *is an open subset of* $\mathbb{C}$, *the complex numbers, and that* $f : G \times [0,1] \to \mathbb{C}$ *is a continuous map such that* $S = \{(t,z) \in [0,1] \times G : f(t,z) = 0\}$ *is compact (possibly empty). If* $f_t(z) = f(t,z)$ *and* f_o *and* f_1 *are complex analytic, then* f_o *and* f_1 *have the same number of zeros in* G *(counting multiplicity).*

PROOF. The homotopy property implies that

$$\deg(f_o, G, 0) = \deg(f_1, G, 0) .$$

It is well-known (and not hard to prove) that for a complex analytic function $g : G \to \mathbb{C}$, $\deg(g, G, 0)$ equals the number of solutions z in G of $g(z) = 0$ (counting multiplicity). □

Of course Lemma 5.1 is precisely Rouché's Theorem if $z \to f_t(z)$ is analytic for $0 \le t \le 1$; otherwise, a little more must be said.

Lemma 5.2. (See [58], [86] and [126]). *Assume that* b *is a nonnegative real and that* c *and* λ *are positive reals. If* $b < c$, *if* $\nu_o \in (0,\pi)$ *is the unique solution of*

$$\cos\nu = -\left(\frac{b}{c}\right), \quad 0 < \nu < \pi, \tag{5.4}$$

and if $\lambda > \frac{\nu_o}{\sqrt{c^2 - b^2}} \equiv \lambda_o$, *then eq.* $(5.3)_\lambda$ *has a unique solution* z *such that*

$\mathrm{Re}(z) > 0$ *and* $0 < \mathrm{Im}\, z < \pi$. *If* $0 \le b < c$ *and* $0 < \lambda < \lambda_0$ *or if* $b \ge c > 0$ *and* $\lambda > 0$, *eq.* $(5.3)_\lambda$ *has no solution* z *such that* $\mathrm{Re}(z) \ge 0$. *If* $0 \le b < c$ *and* $\lambda = \lambda_0$, $\pm i\nu_0$ *are the only solutions of eq.* $(5.3)_\lambda$ *such that* $\mathrm{Re}(z) \ge 0$. *For any* $\lambda > 0$, *eq.* $(5.3)_\lambda$ *has exactly two or zero pure imaginary solutions.*

PROOF. If $0 < c \le b$, $\lambda > 0$ and z is a solution of eq. $(5.3)_\lambda$ with $\mathrm{Re}(z) \ge 0$, one has

$$\left|\frac{z}{\lambda b} + 1\right| = \left|\frac{c}{b}\right| \left|e^{-z}\right| . \tag{5.5}$$

Because $\mathrm{Re}(z) \ge 0$ and $\left|\frac{c}{b}\right| \le 1$, the right hand side of eq. (5.5) is less than or equal to one. Similarly, because $\mathrm{Re}(\frac{z}{\lambda b}) \ge 0$, the left hand side is greater than or equal to one, with equality if and only if $z = 0$. One can see directly that $z = 0$ is not a solution of eq. $(5.3)_\lambda$, so eq. $(5.3)_\lambda$ has no solution z with $\mathrm{Re}(z) \ge 0$ if $0 < c \le b$ and $\lambda > 0$.

If $\pm i\nu$ are pure imaginary solutions of eq. $(5.3)_\lambda$ (so $c > b \ge 0$) one must prove that eq. $(5.3)_\lambda$ has no other solutions $\pm i\nu_1$ with $|\nu_1| \ne |\nu|$. Substituting $z = i\nu$ in eq. $(5.3)_\lambda$ and taking the real and imaginary parts gives

$$\cos \nu = -\frac{b}{c} \tag{5.6}$$

and

$$\lambda = \frac{\nu}{\sin \nu} . \tag{5.7}$$

If $\pm\alpha$ are the unique solutions of eq. (5.6) in $(0,\pi)$ and $(-\pi,0)$ respectively, then the general solution of eq. (5.6) is $\nu = \pm\alpha + 2m\pi$, m an integer. Similarly, if $i\nu_1$ satisfies eq. $(5.3)_\lambda$ one has $\nu_1 = \pm\alpha + 2n\pi$, n an integer. If $\nu = \pm\alpha + 2m\pi$ and $\nu_1 = \pm\alpha + 2n\pi$ (i.e., the same sign in front of α), eq. (5.7) gives

(5.8) $$\lambda = \frac{\pm\alpha+2m\pi}{\pm\sin\alpha} = \frac{\pm\alpha+2n\pi}{\pm\sin\alpha}$$

and $\nu = \nu_1$. If $\nu = \pm\alpha + 2m\pi$ and $\nu_1 = \mp\alpha + 2n\pi$, eq. (5.7) gives

(5.9) $$\lambda = \frac{\pm\alpha+2m\pi}{\pm\sin\alpha} = \frac{\mp\alpha+2n\pi}{\mp\sin\alpha} ,$$

so $(\pm\alpha+2m\pi) = \nu = -(\pm\alpha-2n\pi) = -\nu_1$.

It remains to consider the case $0 \le b < c$ and prove the assertions of the lemma. Define $G = \{z\in\mathbb{C} : \mathrm{Re}(z)>0 \text{ and } 0<\mathrm{Im}(z)<\pi\}$. If

(5.10) $$z + \lambda b + \lambda c e^{-z} = 0$$

for $z = i\nu$ and $0 \le \nu \le \pi$, a simple calculation shows that $\nu = \nu_0$ and $\lambda = \dfrac{\nu_0}{c \sin\nu_0} = \lambda_0$. For future reference note that

(5.11) $$\left(\frac{\nu_0}{\lambda_0 c}\right)^2 + \left(\frac{b^2}{c^2}\right) = \sin^2(\nu_0) + \cos^2(\nu_0) = 1 .$$

For fixed b , c and λ with $b < c$ and $\lambda > \lambda_0$, consider the homotopy (for $z\in\overline{G}$ and $0\le t\le 1$)

(5.12) $$h_t(z) = (1-t)z + t(i\nu_0) + \lambda b + \lambda c e^{-z} .$$

It is easy to see that there exists a constant M such that if $h_t(z) = 0$ for $z \in \overline{G}$, $0 \le t \le 1$, then $\mathrm{Re}(z) \le M$. Furthermore one can easily check that $h_t(z) \ne 0$ for $\mathrm{Re}(z) \ge 0$ and $\mathrm{Im}(z) = 0$ or $\mathrm{Im}(z) = \pi$. If one proves that $h_t(z) \ne 0$ for $z = i\nu$ and $0 \le \nu \le \pi$, Lemma 4.1 will imply that $h_0(z)$ and $h_1(z)$ have precisely the same number of zeros in G . However, if $h_t(i\nu) = 0$ for $0 < \nu < \pi$, then

$$\mathrm{Re}(h_t(i\nu)) = 0 = \lambda b + \lambda c \cos\nu ,$$

so $\nu = \nu_o$. But then

$$\mathrm{Im}(h_t(i\nu)) = \nu_o - \lambda c \sin \nu_o = 0 ,$$

contradicting the assumption that $\lambda \neq \lambda_o$.

To complete the proof that $h_o(z)$ has precisely one zero in G it suffices to prove that $h_1(z)$ has precisely one solution in G . If $h_1(z) = 0$ for $z = x + iy$, then taking absolute values gives

$$\lambda c|e^{-z}| = \lambda c e^{-x} = \sqrt{\nu_o^2+\lambda^2 b^2} \text{ , or}$$

$$e^{-2x} = \left(\frac{\nu_o}{\lambda c}\right)^2 + \left(\frac{b^2}{c^2}\right) . \tag{5.13}$$

Equation (5.13) shows that x is uniquely determined, and eq. (5.11) shows that $x > 0$ if $\lambda > \lambda_o$ and $x < 0$ if $0 < \lambda < \lambda_o$. Taking the imaginary part of $h_1(z) = 0$ and using eq. (5.13) gives

$$\sin(y) = \frac{\nu_o}{\sqrt{\nu_o^2+\lambda^2 b^2}} \tag{5.14}$$

which has a unique solution y with $0 < y < \pi$. Thus, if $\lambda > \lambda_o$, $h_1(z)$ has a unique solution in G .

It remains to prove that $h_o(z)$ has no solutions in $\{z : \mathrm{Re}(z) \geq 0\}$ if $0 < b < c$ and $0 < \lambda < \lambda_o$. Because $h_o(z) = 0$ if and only if $h_o(\bar{z}) = 0$, it suffices to prove that $h_o(z) = 0$ has no solutions in the closure of $H = \{z : \mathrm{Re}(z)>0 \text{ and } \mathrm{Im}(z)>0\}$. To prove the latter statement, it suffices to prove that $h_o(z) = 0$ has no solutions in the closure of $H_N = \{z: \mathrm{Re}(z)>0 \text{ and } 0<\mathrm{Im}(z)<N\pi\}$ for every positive integer N .

If $h_t(z)$ is given as in eq. (5.11), Rouché's theorem shows it suffices to prove that $h_t(z) \neq 0$ for $z \in \partial H_N$ and $0 \leq t \leq 1$ and that there exists a constant M such that every solution $z \in H_N$ of $h_t(z) = 0$ satisfies $Re(z) \leq M$. (The previous argument proved that $h_1(z) \neq 0$ for $Re(z) \geq 0$ and $0<\lambda<\lambda_o$.)

To prove the existence of M, suppose that $h_t(z) = 0$ for $z = x + iy$, $x \geq 0$. Taking the modulus gives

$$\lambda^2 c^2 e^{-2x} = ((1-t)x+\lambda b)^2 + ((1-t)y+t\nu_o)^2 . \tag{5.15}$$

If $0 \leq t \leq \frac{1}{2}$, eq. (5.15) implies that

$$\lambda c e^{-x} \geq (\frac{x}{2}) ,$$

while if $\frac{1}{2} \leq t \leq 1$,

$$\lambda c e^{-x} \geq (\frac{\nu_o}{2}) .$$

Thus one obtains

$$\lambda c e^{-x} \geq (\frac{1}{2}) \min(\nu_o, x) , \tag{5.16}$$

which implies an upper bound on x. It remains to prove that $h_t(z) \neq 0$ for $z \in \partial H_N$. If z is a nonnegative real, this is obvious; if $z = \mu + iN\pi$, it is also easy to see that $Im(h_t(z)) \neq 0$ for $0 \leq t \leq 1$. Finally, if $z = i\nu$ for $0 \leq \nu \leq N\pi$ and $h_t(z) = 0$, one obtains by taking real and imaginary parts that

$$\cos \nu = -\frac{b}{c} \tag{5.17}$$

and

$$\lambda = \frac{(1-t)\nu + t\nu_o}{\sin \nu} . \tag{5.18}$$

The general solution of eq. (5.17) is $\nu = \pm\nu_o + 2m\pi$, m an integer, but because

$\nu > 0$ one must have

$$\nu = \nu_o + 2m\pi \ , \ m \geq 0 \quad \text{or}$$

(5.19) $$\nu = -\nu_o + 2m\pi \ , \ m \geq 1 \ .$$

If ν is as in eq. (5.19), $\sin\nu = -\sin\nu_o$ and eq. (5.18) implies that $\lambda < 0$, a contradiction. Thus one must have $\nu = \nu_o + 2m\pi$ and eq. (5.18) gives

$$\lambda = \frac{\nu_o + (1-t)(2m\pi)}{\sin\nu_o} \geq \lambda_o \ ,$$

which contradicts the assumption that $0 < \lambda < \lambda_o$. □

Consider now the initial value problem related to equation $(5.1)_\lambda$:

$(5.20)_\lambda$ $$\dot{x}(t) = -\lambda g(x(t)) - \lambda f(x(t-1)) \quad \text{for} \quad t \geq 1$$
$$x|[0,1] = \varphi \in C[0,1] \ .$$

The function φ in eq. (5.20) is given (the "initial data") and one wants to solve $(5.20)_\lambda$ for $t \geq 1$. If f is a continuous function and g is locally Lipschitzian, then on the interval [1,2] eq. $(5.20)_\lambda$ reduces to the O.D.E.

$(5.21)_\lambda$ $$\dot{x}(t) = -\lambda g(x(t)) - \lambda f(\varphi(t-1)) \ , \ 1 \leq t \leq 2$$
$$x(1) = \varphi(1) \ .$$

This O.D.E. has a unique solution, and if the solution is defined on the interval [1,2], one can continue the procedure on [2,3], etc. In this way one finds that eq. $(5.20)_\lambda$ has a unique solution

$$x(t) = x(t;\lambda,\varphi)$$

defined on some maximal interval $[0,T)$, $T = T(\lambda,\varphi)$. In the application here, it will be true that $T = \infty$. Furthermore, the map $(\lambda,\varphi) \to x(t;\lambda,\varphi)$ is continuous, uniformly for t in any compact interval $J_0 \subset [0,T)$. Of course, $x(t;\lambda,\varphi)$ and $T(\lambda,\varphi)$ also depend on f and g, but f and g will usually be fixed.

If one considers the initial value problem for the linear equation $(5.2)_\lambda$ one obtains

$$(5.22)_\lambda \qquad \begin{aligned} \dot{u}(t) &= -\lambda b u(t) - \lambda c u(t-1) \\ u|[0,1] &= \varphi \in C[0,1] . \end{aligned}$$

Let $u(t;\lambda,\varphi)$ denote the solution of eq. $(5.22)_\lambda$; it is defined for all $t \geq 0$. If $0 \leq c \leq b$ or $0 \leq b < c$ and $0 < \lambda < \lambda_0$, then Lemma 4.2, in conjunction with the standard theory of linear functional differential equations (see [59], for example), implies

$$\lim_{t\to\infty} u(t;\lambda,\varphi) = 0 .$$

If $g'(0) = b$ and $f'(0) = c$ then one also obtains by standard methods in the theory of functional differential equations (e.g., use the variation of constants formula for F.D.E.'s together with the results for the linearization at 0) a corresponding local result for eq. $(5.20)_\lambda$. This sort of result is basically true whenever the linearization of an F.D.E. has a characteristic equation all of whose roots have negative real parts.

The proof of the following lemma will thus be omitted. The reader is referred to [59] for more details.

LEMMA 5.3. *Assume that* $g : \mathbf{R} \to \mathbf{R}$ *is locally Lipschitzian* $g(0) = 0$ *and* $g'(0) = b \geq 0$ *and that* $f : \mathbf{R} \to \mathbf{R}$ *is continuous with* $g(0) = 0$ *and* $f'(0) = c > 0$. *Suppose that* $0 < c \leq b$, *and* $\lambda > 0$ *or that* $b < c$ *and*

$0 < \lambda < \lambda_0$, where λ_0 is as in Lemma 4.2. There exists $\varepsilon_0 > 0$ such that if $\|\varphi\| < \varepsilon_0$ for any $\varphi \in C[0,1]$ and $x(t;\lambda,\varphi)$ is the corresponding solution of eq. $(5.21)_\lambda$, then $\lim_{t\to\infty} x(t;\lambda,\varphi) = 0$. Furthermore, given any $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$\sup_{t\geq t_\varepsilon} |x(t;\lambda,\varphi)| < \varepsilon$$

for any $\varphi \in C[0,1]$ with $\|\varphi\| < \varepsilon_0$; and there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\sup_{t\geq 0} |x(t;\lambda,\varphi)| < \varepsilon$$

for any $\varphi \in C[0,1]$ with $\|\varphi\| < \delta$. These estimates can be made uniform in λ for λ in a compact interval $I \subset (0,\lambda_0)$.

It is convenient to write down at this point the various hypotheses on f and g which will eventually be needed. Assume that $g : \mathbf{R} \to \mathbf{R}$ is locally Lipschitzian, $g(0) = 0$, g is strictly monotonic increasing. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is continuous and differentiable at 0 with $f'(0) = c > b$ and $f(0) = 0$. Finally, suppose that there exist positive numbers A and B such that

$$-f([-B,A]) = \{-f(x) : -B\leq x\leq A\} \subset g([-B,A]) = [g(-B),g(A)] \tag{5.23}$$

and that

$$xf(x) > 0 \quad \text{for} \quad -B\leq x\leq A \ , \ x \neq 0 \ . \tag{5.24}$$

DEFINITION 5.1. If f and g satisfy the assumptions of the previous paragraph, we shall say that f and g satisfy A1 .

The letters A , B , b and c will be used in the sense of the previous paragraph for the remainder of the discussion of eq. $(5.1)_\lambda$. *Furthermore, it* *will always be assumed that the definition of* f *is modified as follows:*

$$f(x) = \begin{cases} f(A) & \text{for } x \geq A \\ f(x) & \text{for } -B \leq x \leq A \\ f(-B) & \text{for } x \leq -B \end{cases} \tag{5.25}$$

The next lemma gives the motivation for modifying the definition of f as in eq. (5.25).

LEMMA 5.4. *Assume that* f *and* g *satisfy* A1 *and that* f *is modified as in eq.* (5.25). *If* $\varphi \in C[0,1]$, $x(t;\lambda,\varphi)$ *satisfies eq.* $(5.20)_\lambda$ $(\lambda>0)$, *and* $-B \leq x(t_*;\lambda,\varphi) \leq A$ *for some* $t_* \geq 1$, *then*

$$-B \leq x(t;\lambda,\varphi) \leq A \quad \textit{for all} \quad t \geq t_* . \tag{5.26}$$

If $x(t)$ *is any slowly oscillating periodic solution of eq.* $(5.1)_\lambda$, *then* $-B \leq x(t) \leq A$ *for all* t . *If one has*

$$g(-B) < -f(x) < g(A) \quad \textit{for} \quad -B \leq x \leq A , \tag{5.27}$$

and if $-B < x(t_*;\lambda,\varphi)$ *for some* $t_* \geq 1$, *then*

$$-B < x(t;\lambda,\varphi) < A \quad \textit{for all} \quad t \geq t_* . \tag{5.28}$$

In particular, if eq. (5.27) *is satisfied and* $x(t)$ *is a slowly oscillating periodic solution of eq.* $(5.1)_\lambda$, *then*

$$-B < x(t) < A \quad \textit{for all} \quad t . \tag{5.29}$$

PROOF. If the first part of the lemma is false, define $t_0 = \sup\{t \geq t_* : -B \leq x(s;\lambda,\varphi) \leq A \text{ for } t_* \leq s \leq t\}$. One must have $x(t_0)$ equal to A or $-B$, so assume for definiteness that $x(t_0) = A$. By definition of t_0 there exist points $t > t_0$, arbitrarily close to t_0 , such that $x(t) > A$. The mean value theorem then implies that there exists $s > t_0$ such that $\dot{x}(s) > 0$ and $x(s) \geq A$. On the other hand, assumption A1 and eq. (5.25) imply that for this s, $-\lambda f(x(s-1)) \leq \lambda g(A)$ and $-g(x(s)) \leq -\lambda g(A)$, so

$$\dot{x}(s) = -\lambda g(x(s)) + \lambda f(x(s-1)) \leq -\lambda g(A) + \lambda g(A) = 0 ,$$

a contradiction. It follows that $-B \leq x(t) \leq A$ for all $t \geq t_*$.

If one strengthens A1 by assuming that eq. (5.27) holds, if $-B < x(t_*;\lambda,\varphi) < A$ and if $x(t;\lambda,\varphi)$ equals A or $-B$ for some $t > t_*$, define t_0 by

$$t_0 = \inf\{t>t_* : x(t;\lambda,\varphi)=A \text{ or } x(t;\lambda,\varphi)=-B\} .$$

For definiteness assume $x(t_0) = A$, so $\dot{x}(t_0) \geq 0$. On the other hand, eq. (5.27) gives

$$\dot{x}(t_0) = -\lambda g(x(t_0)) + \lambda f(x(t_0-1)) < -\lambda g(A) + \lambda g(A) = 0 ,$$

a contradiction.

If $x(t)$ is a slowly oscillating periodic solution of eq. $(5.1)_\lambda$ and one defines $\varphi = x|[0,1]$, the previous results imply that

$$-B \leq x(t) = x(t;\lambda,\varphi) \leq A \quad \text{for} \quad t \geq z_1 ,$$

where z_1 is the first zero of $x(t)$, with strict inequality if eq. (5.27) is satisfied. Because of periodicity, the estimates on $x(t)$ are valid for all t. □

REMARK 5.1. If one assumes that f and g satisfy A1 with f defined as in eq. (5.25) and if

$$\frac{g(u)}{u} \geq -\frac{g(-B)}{B} \quad \text{for} \quad -B \leq u \leq A , \tag{5.30}$$

then one can prove that any slowly oscillating periodic solution $x(t)$ of eq. $(5.1)_\lambda$ satisfies

$$x(t) > -B \quad \text{for all} \quad t .$$

Similarly, if one has

$$\frac{g(u)}{u} \geq \frac{g(A)}{A} \quad \text{for} \quad -B \leq u \leq A , \tag{5.31}$$

one obtains

$$x(t) < A \quad \text{for all} \quad t .$$

In particular inequality (5.29) is satisfied if $g(u) = u$, even if the strict inequality (5.27) does not hold. These facts follow from the estimates

$$\frac{g(-B)}{u(t)} \int_0^t \lambda u(s)ds < x(t) < \frac{g(A)}{u(t)} \int_0^t \lambda u(s)ds \quad \text{for} \quad t > 0 ,$$

$$u(t) \equiv \exp \int_0^t \lambda \frac{g(x(s))}{x(s)} ds ,$$

but a detailed proof is omitted.

REMARK 5.2. Lemma 5.4 shows that if one is only interested in slowly oscillating periodic solutions $x(t)$ of eq. $(5.1)_\lambda$ such that $-B \leq x(t) \leq A$ for all t , then one may as well assume that f is given by eq. (5.25). However, it may happen that the original eq. $(5.1)_\lambda$ with unmodified f also has other slowly oscillating periodic solutions.

We wish to find an unbounded continuum of slowly oscillating periodic solution of eq. $(5.1)_\lambda$ by using Corollary 4.1. Some definitions will be useful. Define Y to be the Banach space $C[0,1]$ of continuous, real-valued functions on $[0,1]$ in the sup norm and let $K \subset Y$ be the cone

(5.32) $$K = \{\varphi\in Y : \varphi(0)=0 , \varphi(t)\geq 0 \quad \text{for} \quad 0\leq t\leq 1\} .$$

Let $d > 0$ be a constant such that

(5.33) $$|g(y)| \leq d|y| \quad \text{for} \quad -B \leq y \leq A .$$

The letter d will be used in this sense for the remainder of this section. For $\lambda > 0$, define a cone K_λ by

(5.34) $$K_\lambda = \{\varphi\in K : e^{d\lambda t}\varphi(t) \quad \text{is monotonic increasing}\} .$$

If $\varphi \in K-\{0\}$ and $x(t;\lambda,\varphi)$ is the solution of eq. $(5.20)_\lambda$, define reals $z_1(\lambda,\varphi)$ and $z_1^*(\lambda,\varphi)$ by

(5.35) $$z_1(\lambda,\varphi) = \inf\{t\geq 1 : x(t;\lambda,\varphi)=0\} \quad \text{and}$$

(5.36) $$z_1^*(\lambda,\varphi) = \sup\{t\geq 1 : x(s;\lambda,\varphi)\geq 0 \quad \text{for} \quad 1\leq s\leq t\} .$$

If $x(t;\lambda,\varphi) > 0$ for all $t \geq 1$, define $z_1(\lambda,\varphi) = \infty$. If $z_1(\lambda,\varphi) < \infty$ we shall shortly see that

$$z_1^*(\lambda,\varphi) < z_1(\lambda,\varphi) + 1 ,$$

so it makes sense to define

(5.37) $$z_2(\lambda,\varphi) = \inf\{t>z_1^*(\lambda,\varphi) : x(t;\lambda,\varphi)=0\} .$$

As before, if $x(t;\lambda,\varphi) < 0$ for $t > z_1^*(\lambda,\varphi)$, one defines $z_2(\lambda,\varphi) = +\infty$. Notice that $\varphi = 0$ is not allowed.

Suppose now that $\varphi \in K-\{0\}$, that $x(\zeta;\lambda,\varphi) = 0$ and that $x(t;\lambda,\varphi) \geq 0$ for $1 \leq t \leq \zeta$. For convenience, write $x(t) = x(t;\lambda,\varphi)$. The first claim is that $x(t) \leq 0$ for $\zeta \leq t \leq \zeta + 1$. If not, the mean value theorem implies that there exists t, $\zeta < t < \zeta + 1$, such that $x(t) > 0$ and $\dot{x}(t) > 0$. However,

$$\dot{x}(t) = -\lambda g(x(t)) - \lambda f(x(t-1)) < 0 ,$$

(because $g(x(t)) > 0$ and $f(x(t-1)) \geq 0$) , a contradiction.

The next observation is that $e^{d\lambda(t-\zeta)}x(t)$ is monotonic decreasing for $\zeta \leq g \leq \zeta + 1$. To see this observe that

$$\dot{x}(t) + \lambda dx(t) \leq \dot{x}(t) + \lambda g(x(t)) = -\lambda f(x(t-1)) \leq 0 , \zeta \leq t \leq \zeta + 1 ,$$

multiply by $e^{d\lambda(t-\zeta)}x(t)$, and integrate.

If one uses these observations one sees that $x(t) = 0$ for $z_1(\lambda,\varphi) \leq t \leq z_1^*(\lambda,\varphi)$, $x(t) > 0$ for $z_1^*(\lambda,\varphi) < t \leq z_1^*(\lambda,\varphi) + 1$ and $e^{d\lambda(t-z_1^*)}x(t)$ is monotonic decreasing on $z_1^* \leq t \leq z_1^* + 1$. If $z_1^* \geq z_1 + 1$, one would have

$$\dot{x}(t) = 0 = -\lambda f(x(t-1)) , z_1 \leq t \leq z_1 + 1 ,$$

and the previous equation would imply $x(s) = 0$ for $z_1 - 1 \leq s \leq z_1$. If $z_1 > 1$, this contradicts the definition of z_1 ; while if $z_1 = 0$, this implies $\varphi = 0$, a contradiction. Thus one concludes that $z_1^* < z_1 + 1$.

The previous observations show that $z_2(\lambda,\varphi) > z_1^*(\lambda,\varphi) + 1$. Equation $(5.20)_\lambda$ directly implies that $\dot{x}(t) \leq 0$ for $1 \leq t \leq z_1^*(\lambda,\varphi)$ and (because $x(t)<0$ for $z_1^*(\lambda,\varphi) < t \leq z_1^*(\lambda,\varphi) + 1$) $\dot{x}(t) > 0$ for $z_1^*(\lambda,\varphi) + 1 < t \leq z_2(\lambda,\varphi)$.

The same kind of argument used before shows that $x(t) \geq 0$ for $z_2 \leq t \leq z_2 + 1$ and that $e^{\lambda d(t-z_2)}x(t)$ is monotonic increasing and positive at $z_2 + 1$. If one had $x(t) = 0$ for some t with $z_2 < t < z_2 + 1$, eq. $(5.20)_\lambda$ would imply $\dot{x}(t) > 0$, a contradiction, so $x(t) > 0$ for $z_2 < t \leq z_2 + 1$.

It is not hard to see that the map $(\lambda,\varphi) \to z_1(\lambda,\varphi)$ $(\lambda>0\ ,\ \varphi\in K-\{0\})$ need not be continuous. However, one has

LEMMA 5.5. *Assume that* f *and* g *satisfy A1. The map* $(\lambda,\varphi) \to z_2(\lambda,\varphi)$ *is continuous as a map from* $(0,\infty) \times (K-\{0\})$ *to* $[1,\infty) \cup \{\infty\}$.

PROOF. Suppose that $\lambda > 0$ and $\varphi \in K-\{0\}$. If $(\lambda_n,\varphi_n) \to (\lambda,\varphi)$, one must prove that $z_2(\lambda_n,\varphi_n) \to z_2(\lambda,\varphi)$. If $z_1(\lambda,\varphi) = \infty$, $x(t;\lambda,\varphi)$ is positive on $[1,\infty)$. Given any $T > 0$, it follows by continuous dependence on initial data that $x(t;\lambda_n,\varphi_n)$ is positive on $[1,T]$ for n large enough, and this implies $z_2(\lambda_n,\varphi_n) \to +\infty$.

Thus assume that $z_1(\lambda,\varphi) < \infty$ and $z_1(\lambda,\varphi) > 1$ (the proof if $z_1(\lambda,\varphi)=1$ is analogous but easier). Select η , $0 < \eta < z_1$, so that if $t_o = z_1(\lambda,\varphi) - \eta$ and $t_1 = z_1^*(\lambda,\varphi) + \eta$, then $x(t;\lambda,\eta) \ge 2\alpha > 0$ for $1 \le t \le t_o$, $x(t;\lambda,\varphi) \le -2\alpha$ for $t_1 \le t \le t_1 + 1$ and $t_1 - t_o < 1$. For m large enough one has $x(t;\lambda_m,\varphi_m) \ge \alpha$ for $1 \le t \le t_o$, and $x(t;\lambda_m,\varphi_m) \le -\alpha$ for $t_1 \le t \le t_1 + 1$. One concludes that for m large enough

$$t_o < z_1(\lambda_m,\varphi_m) \le z_1^*(\lambda_m,\varphi_m) < t_1 .$$

Because $z_2(\lambda_m,\varphi_m) - z_1^*(\lambda_m,\varphi_m) > 1$ and $t_1 - t_o < 1$, one must have $z_2(\lambda_m,\varphi_m) > t_1 + 1$ for m large enough.

If $\varepsilon > 0$ is sufficiently small and $z_2(\lambda,\varphi) < \infty$, then because

$$\left.\frac{d}{dt} x(t;\lambda,\varphi)\right|_{t=z_2(\lambda,\varphi)} > 0 ,$$

one has $x(z_2+\varepsilon;\lambda,\varphi) > 0$ and $x(t;\lambda,\varphi) < 0$ for $t_1 + 1 \le t \le z_2 - \varepsilon$, where $z_2 = z_2(\lambda,\varphi)$. For m large enough, continuous dependence on initial data implies that the same inequalities are satisfied by $x(t;\lambda_m,\varphi_m)$, and one concludes that

(for m large enough)

$$z_2(\lambda,\varphi) - \varepsilon < z_2(\lambda_m,\varphi_m) < z_2(\lambda,\varphi) + \varepsilon .$$

It remains only to prove continuity when $z_1(\lambda,\varphi) < \infty$ and $z_2(\lambda,\varphi) = \infty$. In this case $x(t;\lambda,\varphi) < 0$ for $t \geq t_1$. If $T > t_1$, one has for m large enough

$$x(t;\lambda_m,\varphi_m) < 0 \quad \text{for} \quad t_1 \leq t \leq T ,$$

so $z_2(\lambda_m,\varphi_m) > T$ for m large enough and $\lim_{m\to\infty} z_2(\lambda_m,\varphi_m) = \infty$. □

Another technical lemma is, unfortunately, necessary for the subsequent work. The following result is a direct consequence of eq. $(5.20)_\lambda$ and is left to the reader.

LEMMA 5.6. *Assume that* f *and* g *satisfy A1 and that, for* $\lambda < 0$ *and* $\varphi \in K\text{-}\{0\}$, $x(t) = x(t;\lambda,\varphi)$ *is the solution of eq.* $(5.20)_\lambda$. *The function* $x(t)$ *is monotonic decreasing on* $[1,z_1^*(\lambda,\varphi)]$ *and monotonic increasing on* $[z_1^*+1,z_2]$. *Define a monotonic increasing function* $h(u)$ *by*

$$h(u) = \begin{cases} \max\{f(s): 0 \leq s \leq u\} & \text{for } u \geq 0 \\ \min\{f(s): u \leq s \leq 0\} & \text{for } u \leq 0 . \end{cases}$$

If, for $\tau \leq z_1^* - 1$ *one defines* $M = \max\{x(t): \tau \leq t \leq \tau+1\} \leq \max\{x(t) : 0\leq t\leq 1\}$, *one obtains*

$$\min\{x(t) : z_1^*\leq t\leq z_1^*+1\} \geq -\lambda h(M) .$$

Similarly, if for $\delta \leq z_2 - 1$ *one defines* $M_1 = \min\{x(t) : \delta\leq t\leq \delta+1\}$ *one obtains*

$$\max\{x(t) : z_2\leq t\leq z_2+1\} \leq -\lambda h(M_1) .$$

Using the previous results I want to describe a compact, continuous map $\Psi : (0,\infty) \times K \to K$. Assume that f and g satisfy A1. If $(\lambda,\varphi) \in (0,\infty) \times (K-\{0\})$ and $z_2(\lambda,\varphi) = z_2 < \infty$, let $x(t;\lambda,\varphi)$ denote the solution of eq. $(5.20)_\lambda$ and define Ψ by

$$\Psi(\lambda,\varphi) = \psi$$

where

$$\psi(t) = x(z_2+t;\lambda,\varphi) \ , \ 0 \le t \le 1 \ , \ z_2 = z_2(\lambda,\varphi) \ . \tag{5.38}$$

If $z_2(\lambda,\varphi) = \infty$ or $\varphi = 0$, define

$$\Psi(\lambda,\varphi) = 0 \ . \tag{5.39}$$

LEMMA 5.7. *Assume that* f *and* g *satisfy* A1 *and that* f *is extended as in eq.* (5.25). *If* Ψ *is defined as above,* Ψ *is a continuous, compact map of* $(0,\infty) \times K \to K$ *and* $\|\Psi(\lambda,\varphi)\| \le A$ *for all* (λ,φ) . *If* $\Psi_\lambda(\varphi) \equiv \Psi(\lambda,\varphi)$, $\Psi_\lambda(K) \subset K_\lambda$ *and* $\Psi_\lambda(\varphi) = \varphi$ *for* $\varphi \in K-\{0\}$ *implies that* $x(t;\lambda,\varphi)$ *extends to a slowly oscillating periodic solution* $x(t)$ *of eq.* $(5.1)_\lambda$ *such that* $-B \le x(t) \le A$ *for all* t.

PROOF. Lemma 5.4 implies that

$$-B \le x(t;\lambda,\varphi) \le A$$

for $t \ge z_1(\lambda,\varphi)$, and eq. $(5.20)_\lambda$ then implies that for λ in a bounded interval, there exists a constant M such that

$$\left|\frac{d}{dt} x(t;\lambda,\varphi)\right| \le M \ , \ t \ge z_1(\lambda,\varphi) \ ,$$

where M is independent of $\varphi \in K$. The Ascoli-Arzela theorem thus implies that $\{\Psi(\lambda,\varphi) : \lambda>0, \lambda \text{ bounded}, \varphi\in K\}$ has compact closure.

The fact that $\|\Psi(\lambda,\varphi)\| \le A$ follows from Lemma 5.4, and the inclusion $\Psi_\lambda(K) \subset K_\lambda$ was proved in the remarks preceding Lemma 5.5. If $\Psi(\lambda,\varphi) = \varphi$ for $\varphi \in K-\{0\}$ and one extends $x(t;\lambda,\varphi)$ to be a periodic function $x(t)$ of period $z_2(\lambda,\varphi)$, it is left to the reader to check that $x(t)$ satisfies eq. $(5.1)_\lambda$ for all t and is a slowly oscillating periodic solution.

It remains to prove that Ψ is continuous. Suppose $(\lambda,\varphi) \in (0,\infty) \times K$ and that $(\lambda_n,\varphi_n) \to (\lambda,\varphi)$. If $\varphi \ne 0$ and $z_2(\lambda,\varphi) < \infty$, Lemma 5.5 implies that $z_2(\lambda_n,\varphi_n) \to z_2(\lambda,\varphi)$ and continuous dependence on initial data implies (writing $\zeta=z_2(\lambda,\varphi)$)

$$\sup\{|x(t;\lambda,\varphi)-x(t;\lambda_n,\varphi_n)| : \zeta\le t\le\zeta+1\} \to 0 . \tag{5.40}$$

If one writes $\zeta_n = z_2(\lambda_n,\varphi_n)$ and $x_n(t) = x(t;\lambda_n,\varphi_n)$, one has a uniform bound on $|\frac{d}{dt}x_n(t)|$ for $t \ge z_1(\lambda_n,\varphi_n)$, and this yields

$$\lim_{n\to\infty} \sup\{|x_n(\zeta_n+s)-x_n(\zeta+s)| : 0\le s\le 1\} = 0 . \tag{5.41}$$

Equations (5.40) and (5.41) imply

$$\lim_{n\to\infty} \Psi(\lambda_n,\varphi_n) = \Psi(\lambda,\varphi) .$$

If $\varphi \ne 0$ and $z_2(\lambda,\varphi) = +\infty$, Lemma 5.5 implies that $\lim_{n\to\infty} \zeta_n = \infty$; and it has already been remarked that

$$\lim_{t\to\infty} \sup|x(t)| = 0 .$$

Select T so large that

$$\sup\{|x(t)| : T\le t\le T+2\} < \varepsilon .$$

For n large enough one can assume that $T + 2 \le \zeta_n$ and

$$\sup\{|x_n(t)| : T \le t \le T+2\} < 2\varepsilon .$$

Lemma 5.6 now implies that for n large enough

$$\|\Psi(\lambda_n, \varphi_n)\| \le \max(-\lambda_n h(-\varepsilon), -\lambda_n h(-\lambda_n h(\varepsilon))) ,$$

where h is as in Lemma 5.6; and this estimate gives $\lim_{n\to\infty} \Psi(\lambda_n, \varphi_n) = 0$.

It remains only to consider the case $\varphi = 0$. If $\psi \in K$ and $\mu > 0$, Lemma 5.6 implies

$$\|\Psi(\mu, \psi)\| \le -\mu h(-\mu h(\|\psi\|)) . \tag{5.42}$$

If one applies eq. (5.42) to the case $\mu = \lambda_n$ and $\psi = \varphi_n$ (where $\varphi_n \to 0$) and one recalls that $h(0) = 0$ and h is continuous, one finds

$$\lim_{n\to\infty} \Psi(\lambda_n, \varphi_n) = 0 . \quad \square$$

REMARK 5.3. I am illustrating here a *general* approach to the problem of proving existence *in the large* of periodic solutions of differential-delay equations. As is discussed at greater length in [102], for example, this approach has been applied to many examples. When the general method fails, it often does so because one cannot globally define a map like Ψ .

This elementary difficulty has frequently not been appreciated in the literature, e.g., in some work of G.S. Jones [69]. For this reason I have been fairly careful in discussing Ψ . The reader should realize that even in this "nice" case, Ψ_λ is not Fréchet differentiable at $\varphi = 0$ and that in other examples [96, Section 3] analogues of Ψ can be defined which are not necessarily continuous at 0 .

REMARK 5.4. Because f is defined as in eq. (5.25) and is differentiable at 0 , there exists a constant D such that

(5.43) $$|f(x)| \le D|x| \;;$$

and it follows from this that (for h as in Lemma 5.6)

$$|h(u)| \le D|u| \;.$$

Using this estimate in eq. (5.42), one obtains

(5.44) $$\|\Psi(\mu,\psi)\| \le (\mu D)^2\|\psi\| \;.$$

Eq. (5.43) immediately implies that ψ_λ has no non-zero fixed points for $0 < \lambda \le D^{-1}$, which is a result that will be needed later. If eq. (5.43) fails, it may, in fact, happen that eq. $(5.1)_\lambda$ has a slowly oscillating periodic solution for every $\lambda > 0$: this is the case for $f(x) = x^{1/3}$ and $g(x) = x$, for example.

One more lemma is necessary to prove the main theorem of this section.

LEMMA 5.8. *Assume that* f *and* g *satisfy A1, that* $b = g'(0) < c = f'(0)$ *and that* $\lambda > \lambda_0 = \dfrac{\nu_0}{\sqrt{c^2-b^2}}$, *where* ν_0 *is defined in Lemma 5.2. Then there exists* $\varepsilon = \varepsilon(\lambda) > 0$ *such that if* $\varphi \in K-\{0\}$,

(5.45) $$\limsup_{t\to\infty} |x(t;\lambda,\varphi)| \ge \varepsilon(\lambda) \;.$$

If J_0 *is a compact interval such that* $J_0 \subset (\lambda_0,\infty)$, *there exists* $\varepsilon = \varepsilon(J_0)$ *such that for* $(\lambda,\varphi) \in J_0 \times (K-\{0\})$,

$$\limsup_{t\to\infty} |x(t;\lambda,\varphi)| \ge \varepsilon(J_0) \;.$$

PROOF. Define $\delta_1(x) = \sup\{|g(s)-bs| : |s|\le|x|\}$ and $\delta_2(x) = \sup\{|f(s)-cs| : |s|\le|x|\}$ and observe that

$$\lim_{x\to 0} \frac{\delta_j(x)}{x} = 0 \;.$$

If $\lambda > \lambda_o$, recall that Lemma 5.2 insures the existence of a unique solution $\zeta = \zeta(\lambda) = \mu(\lambda) + i\nu(\lambda) = \mu + i\nu$ of eq. (5.10) such that $\mu \geq 0$ and $0 \leq \nu \leq \pi$ (and, in fact, $\mu>0$ and $0<\nu<\pi$) . One can easily see that $\lambda \to \zeta(\lambda)$ is continuous for $\lambda > \lambda_o$.

If $\lambda > \lambda_o$ and $\varphi \in K-\{0\}$, define $x(t) = x(t;\lambda,\varphi)$ and let z_n , $n \geq 2$, denote the consecutive zeros of $x(t)$ after $z_2 = z_2(\lambda,\varphi)$. The same sort of argument as used before Lemma 5.5 shows that $(-1)^n x(t) > 0$ for $z_n < t < z_{n+1}$ and $(-1)^n \frac{d}{dt} (e^{\lambda dt}x(t)) \geq 0$ for $z_n < t < z_n + 1$ and $(-1)^n \dot{x}(t) \leq 0$ for $z_n + 1 \leq t \leq z_{n+1}$, $n \geq 2$.

A *priori*, it is possible that there are only finitely many zeros z_n , $n \geq 2$; and the first claim is that this is not the case. Suppose, by way of contradiction that $x(t) > 0$ on an interval (T,∞) ; the proof is similar if $x(t) < 0$ on (T,∞) . By using eq. $(5.20)_\lambda$ and the assumption that $uf(u) > 0$ for $u \neq 0$, one finds that $\lim_{t\to\infty} x(t) = 0$. Select c_1 , $0 < c_1 < c$, so that

$$\frac{\nu_o c_1}{\sqrt{c^2-b^2}} = k > 1 .$$

Take $\delta > 0$ so that $|f(u)| \geq c_1|u|$ for $|u| \leq \delta$ and select $T_1 > T + 2$ so that $0 \leq x(t) \leq \delta$ for $T_1 - 1 \leq t$. One can see that $x(t)$ is monotonic decreasing on (T_1-1,∞) . If one defines $\alpha = x(T_1)$ and uses eq. $(5.20)_\lambda$, one sees that for $T_1 \leq t \leq T_1 + 1$ one has

$$\dot{x}(t) \leq -\lambda f(x(t-1)) \leq -\lambda c_1 x(t-1) \leq -\lambda_o c_1 \alpha = -k\alpha .$$

It follows from the previous inequality that

$$x(T_1+1) \leq \alpha - k\alpha < 0 ,$$

a contradiction.

For a given $T \geq z_1(\lambda,\varphi)$, define $\eta(T)$ by

$$\eta(T) = \sup\{|x(t)| : t \geq T\} .$$

It suffices to find $\varepsilon(\lambda) > 0$, $\varepsilon(\lambda)$ independent of $\varphi \in K-\{0\}$, such that $\eta(T) \geq \varepsilon(\lambda)$. For any T , there exists $z_m > T$ such that

$$\sup\{|x(t)| : z_m \leq t \leq z_{m+1}\} > \frac{1}{2}\eta(z_m) . \tag{5.46}$$

Because $\eta(T) \geq \eta(z_m)$, it suffices to find an appropriate lower estimate on $\eta(z_m)$. For notational convenience, write $\eta = \eta(z_m)$ and $\zeta = \zeta(\lambda) = \mu + i\nu$. λ

If $T \geq 1$, integration by parts gives

$$\int_T^\infty \dot{x}(t)e^{-\zeta(t-T)}dt = \zeta\int_T^\infty x(t)e^{-\zeta(t-T)}dt - x(T) . \tag{5.47}$$

On the other hand, substituting for $\dot{x}(t)$ from eq. $(5.20)_\lambda$ and writing $\theta_1(u) = g(u) - bu$ and $\theta_2(u) = f(u) - cu$ gives

$$\begin{aligned}\int_T^\infty \dot{x}(t)e^{-\zeta(t-T)}dt &= -\lambda b\int_T^\infty x(t)e^{-\zeta(t-T)}dt - \lambda ce^{-\zeta}\int_T^\infty x(t)e^{-\zeta(t-T)}dt \\ &\quad - \lambda\int_T^\infty \theta_1(x(t))e^{-\zeta(t-T)}dt - \lambda e^{-\zeta}\int_{T-1}^\infty \theta_2(x(t))e^{-\zeta(t-T)}dt .\end{aligned} \tag{5.48}$$

Equating the right hand sides of equation (5.47) and (5.48) and using eq. (5.10) to simplify yields

$$\begin{aligned}-x(T) + \lambda c\int_{T-1}^T x(t)e^{-\zeta(t-T+1)}dt &= -\lambda\int_T^\infty \theta_1(x(t))e^{-\zeta(t-T)}dt \\ &\quad -\lambda\int_{T-1}^\infty \theta_2(x(t))e^{-\zeta(t-T+1)}dt .\end{aligned} \tag{5.49}$$

If one selects $T = z_m + 1$ in eq. (5.49) and designates the right hand side of eq. (5.49) by R.H.S., one obtains

$$|\text{R.H.S.}| \leq \lambda(\delta_1(\eta)+\delta_2(\eta)) \int_0^\infty |e^{-\zeta s}|dt = \frac{\lambda(\delta_1(\eta)+\delta_2(\lambda))}{\mu} . \tag{5.50}$$

In order to find a lower estimate on the absolute value of the left hand side of eq. (5.49) it is useful to consider subcases. One can assume that $x(t) \geq 0$ on $[T-1,T]$, because the proof is essentially the same if $x(t) \leq 0$ on $[T-1,T]$.

Case 1. $\lambda c \int_{T-1}^{T} x(t)dt \leq \frac{\eta}{4e^{\lambda d}}$. Because $e^{\lambda d(t-z_m)} x(t)$ is monotonic increasing on $[T-1,T]$ and because $\sup\{x(t) : T-1 \leq t \leq T\} \geq (\frac{\eta}{2})$, one obtains

$$x(t) \geq \frac{\eta}{2e^{\lambda d}} .$$

Thus the left hand side of eq. (5.49) has absolute value greater than or equal to

$$x(T) - \left|\lambda c \int_{T-1}^{T} x(t)e^{-\zeta(t-T+1)}dt\right| \geq \frac{\eta}{4e^{\lambda d}} . \tag{5.51}$$

Combining equations (5.50) and (5.51) gives

$$\frac{\mu}{4\lambda e^{\lambda d}} \leq \frac{\delta(\eta)}{\eta} , \tag{5.52}$$

where

$$\delta(\eta) = \delta_1(\lambda) + \delta_2(\eta) . \tag{5.53}$$

Case 2. $\lambda c \int_{T-1}^{T} x(t)dt > \frac{\eta}{4e^{\lambda d}}$. The absolute value of the left hand side of eq. (5.49) is bounded below by the absolute value of its imaginary part, so one obtains (because $0<\nu<\pi$)

$$|\text{L.H.S.}| \geq c\lambda \int_{T-1}^{T} x(t) e^{-\mu(t-T+1)} \sin \nu (t-T+1) dt$$

(5.54)

$$\geq [\min(\sin \nu, \sin(\tfrac{\nu}{2}))](e^{-\mu})(c\lambda \int_{T-\frac{1}{2}}^{T} x(t) dt) \ .$$

Because $e^{\lambda d(t-z_m)} x(t)$ is monotonic increasing for $z_m \leq t \leq x_m + 1$ one obtains

$$x(t + \tfrac{1}{2}) \geq \exp(-\tfrac{\lambda d}{2}) x(t) \quad \text{for} \quad z_m \leq t \leq z_m + \tfrac{1}{2} \ .$$

The previous equation implies

$$\int_{T-\frac{1}{2}}^{T} x(s) ds \geq \exp(-\tfrac{\lambda d}{2}) \int_{T-1}^{T-\frac{1}{2}} x(s) ds \ ,$$

so

$$c\lambda(1 + e^{\frac{\lambda d}{2}})(\int_{T-\frac{1}{2}}^{T} x(s) ds) \geq c\lambda \int_{T-1}^{T} x(s) ds$$

(5.55)

$$\geq \frac{\eta}{4e^{\lambda d}} \ .$$

Combining equations (5.50), (5.54) and (5.55) and simplifying gives

(5.56) $$\frac{\mu \alpha e^{-\mu}}{4\lambda e^{\lambda d}(1+e^{\frac{1}{2}\lambda d})} \leq \frac{\delta(\eta)}{\eta} \ ,$$

where

$$\alpha = \min(\sin \nu, \sin(\tfrac{\nu}{2})) \ .$$

It follows that inequality (5.56) is satisfied in case 1 or case 2. Because $\frac{\delta(\eta)}{\eta}$ approaches 0 as $\eta \to 0$, inequality (5.56) implies that there must be a positive constant $\varepsilon(\lambda) > 0$ such that $\eta \geq \varepsilon(\lambda)$. Because $\mu = \mu(\lambda)$ and $\nu = \nu(\lambda)$ are continuous functions of λ with $\mu > 0$ and $0 < \nu < \pi$, one can find a positive lower bound for the left hand side of eq. (5.56) if λ is restricted to a compact interval $J_o \subset (\lambda_o, \infty)$. Equation (5.46) then implies that there exists a positive constant $\varepsilon(J_o)$ such that $\eta \geq \varepsilon(J_o)$. □

With these preliminaries it is possible to prove the main theorem of this section. Notice that if $x(t)$ is any periodic solution of eq. $(5.1)_\lambda$ (not necessarily slowly oscillating) then one can map the ordered pair (λ,x) to the ordered pair (λ,φ) , where $\varphi = x|[0,1] \in Y$. Using the periodicity of $x(t)$ one can see that this map is one-one. Conversely, given an ordered pair $(\lambda,\varphi) \in J \times Y$ $(J=(0,\infty)$, $Y=C[0,1])$ such that $x(t;\lambda,\varphi) = x(t+p;\lambda,\varphi)$ for all $t \geq 0$ and some $p > 0$ (where $x(t;\lambda,\varphi)$ is the solution of eq. $(5.20)_\lambda$) , one can extend $x(t;\lambda,\varphi)$ by periodicity so it is defined for all t and it will satisfy eq. $(5.1)_\lambda$ for all t . Thus periodic solutions of eq. $(5.1)_\lambda$ can be identified with a subset of $J \times Y$. Under this identification, the set of slowly oscillating periodic solutions is the set $\{(\lambda,\varphi)\in J\times K : \Psi(\lambda,\varphi)=\varphi , \varphi\neq 0\}$. The latter set is not, as will be seen, closed in $J \times K$ so define S by

$$S = \{(\lambda,\varphi)\in J\times K : \Psi(\lambda,\varphi)=\varphi \text{ and } \varphi\neq 0\} \cup \{(\lambda_o,0)\} . \tag{5.57}$$

In eq. (5.57), λ_o is defined as in Lemma 5.2 and it is assumed that $b = g'(0) < c = f'(0)$; of course Ψ is defined by equation (5.38) and (5.39).

THEOREM 5.1. *Assume that* f *and* g *satisfy A1, that* f *satisfies eq. (5.25) and that* $b = g'(0) < c = f'(0)$. *Define* ν_o *to be the unique solution of* $\cos(\nu) = -\frac{b}{c}$ *such that* $\frac{\pi}{2} \leq \nu < \pi$ *and* $\lambda_o = \dfrac{\nu_o}{\sqrt{c^2-b^2}}$. *If* S *is defined by eq. (5.57)*, S *is closed in* $J \times K$. *There exists* $\delta > 0$ *such that* $\lambda \geq \delta$ *for all* $(\lambda,\varphi) \in S$ *and* $-B \leq x(t;\lambda,\varphi) \leq A$ *for all* t *and all* $(\lambda,\varphi) \in S$, *where* $x(t;\lambda,\varphi)$ *is the solution of eq.* $(5.20)_\lambda$ *and* A *and* B *are as in A1. If* S_o *is the connected component of* S *which contains* $(\lambda_o,0)$, *then* S_o *is unbounded and* $\{\lambda : (\lambda,\varphi)\in S_o$ *for some* $\varphi\in K-\{0\}$ *or* $\lambda=\lambda_0\}$ *is an interval which contains* $[\lambda_o,\infty)$. *In particular, for every* $\lambda > \lambda_o$, *eq.* $(5.1)_\lambda$ *has a slowly oscillating periodic* *solution* $x_\lambda(t)$ *such that* $-B \le x_\lambda(t) \le A$ *for all* t .

PROOF. Lemma 5.3 implies that if $J_0 \subset (0,\lambda_0)$ is a compact interval, there exists $\varepsilon > 0$ such that $(\lambda,\varphi) \notin S$ $(\lambda,\varphi) \in J_0 \times K$ and $\|\varphi\| < \varepsilon$. By using Lemmas 5.6 and 5.8 one finds that the same result is true if J_0 is a compact interval $J_0 \subset (\lambda_0,\infty)$. By Remark (5.4), there exists $\delta > 0$ such that $\lambda \ge \delta$ for $(\lambda,\varphi) \in S$. The previous remarks show that if $(\lambda_n,\varphi_n) \in S \to (\lambda,0)$, then $\lambda = \lambda_0$. If $(\lambda_n,\varphi_n) \in S \to (\lambda,\varphi)$, $\varphi \ne 0$, then $\lambda \ge \delta$ and the continuity of Ψ (Lemma 5.7) implies that $\Psi(\lambda,\varphi) = \varphi$. Thus $(\lambda,\varphi) \in S$ and S is closed. The remaining assertions about S are proved in Lemma 5.7 and Remark 5.4.

It remains to prove that S_0 is unbounded, and this will follow from Corollary 4.1. Lemma 5.7 implies that Ψ is a compact, continuous map. By Lemma 5.3, 0 is an attractive fixed point of Ψ_λ for $0 < \lambda < \lambda_0$; and from Lemmas 5.6 and 5.8 one derives that 0 is an ejective fixed point of Ψ_λ for $\lambda > \lambda_0$. The other hypotheses of Corollary 4.1 have already been verified; so S_0 is unbounded. Because $\|\varphi\| \le A$ for all $(\lambda,\varphi) \in S_0$, the set $\{\lambda : (\lambda,\varphi) \in S_0 \text{ for some } \varphi \in K\} = L$ must be an interval (the continuous image of a connected set), must be unbounded and of course contains λ_0 . This shows that $L \supset [\lambda_0,\infty)$, and the last assertion of the theorem follows immediately. □

If one is only interested in the existence of a slowly oscillating periodic solution of eq. $(5.1)_\lambda$ for each $\lambda > \lambda_0$ and if one restricts to the case $g(x) = x$, then one can use a result of Hadeler and Tomiuk [58]. However, I would like to show now that much more information can be obtained from the existence of the unbounded connected set S_0 .

The next lemma is proved in [86] for the case $g(x) = x$, but essentially the same proof works in general. The lemma is not hard (in fact, part of it has been proved in Lemma 5.5) but for reasons of length the proof will be omitted.

LEMMA 5.9. (See [86]). *Assume that* f *and* g *satisfy A1 and that* $g'(0) = b < f'(0) = c$. *If* $(\lambda,\varphi) \in S$ *and* $(\lambda,\varphi) \neq (\lambda_o,0)$, $z_2(\lambda,\varphi)$ *has been defined; define*

$$z_2(\lambda_o,0) = \frac{2\pi}{\nu_o} ,$$

where ν_o *is as in Theorem 5.1. Then the map* $z_2 : S \to \mathbf{R}$ *is continuous. Furthermore, there exists* $Q > 0$ *such that* $2 < z_2(\lambda,\varphi) \leq Q$ *for all* $(\lambda,\varphi) \in S$.

Recall that periodic solutions $x(t)$ of eq. $(5.1)_\lambda$ can be identified with pairs (λ,φ) , $\varphi = x|[0,1] \in C[0,1]$. We allow the possibility that $\lambda < 0$. If hypotheses are as in Theorem 5.1, n is any integer, and $(\lambda,\varphi) \in S_o$, define $x(t) = x(t;\lambda,\varphi)$ to be the slowly oscillating periodic solution corresponding to (λ,φ) (or $\lambda=\lambda_o$, $\varphi=0$ and $x(t)\equiv 0$) and define $y(t)$ by

$$y(t) = x((np+1)t) , \quad p = z_2(\lambda,\varphi) . \tag{5.58}$$

A calculation shows that

$$\begin{aligned} \dot{y}(t) &= -\lambda_1 g(y(t)) - \lambda_1 f(y(t-1)) \\ \lambda_1 &= (np+1)\lambda , \end{aligned} \tag{5.59}$$

so $y(t)$ is a periodic solution of minimal period $\frac{p}{|np+1|}$ of eq. $(5.1)_{\lambda_1}$. Define a map $J_n : S_o \to \mathbf{R} \times Y$ by $J_n(\lambda,\varphi) = (\lambda_1,\psi)$, where $\lambda_1 = \lambda(np+1)$ and $\psi = y|[0,1]$. By using Lemma 5.9, one sees that J_n is continuous, so $S_n = J_n(S_o)$ is an unbounded, connected set of periodic solutions of eq. $(5.1)_\lambda$. It is also true that $-B \leq y(t) \leq A$ for y as above and $(\lambda,0) \in S_n$ if and only if $\lambda = \lambda_n = \frac{\nu_o + 2n\pi}{\sqrt{c^2-b^2}}$. By using information about the minimal period of elements of

S_n (see [86]) it is not hard to prove that $S_n \cap S_m$ is empty for $n \neq m$. One concludes by a simple connectivity argument $\{\lambda : (\lambda,\psi) \in S_n$ for some $\psi\} \supset [\lambda_n, \infty)$ if $n \geq 0$ and contains $(-\infty, \lambda_n]$ if $n \leq -1$, and that consequently eq. $(5.1)_\lambda$ has at least $n + 1$ distinct periodic solutions (all vanishing at $t=0$) for each $\lambda > \lambda_n$ and n a nonnegative integer, and eq. $(5.1)_\lambda$ has at least $|n|$ distinct periodic solutions for each $\lambda < \lambda_n$ and n a negative integer.

It is interesting to note that one can also prove that J_n is one-one for all n, but I omit the proof for reasons of length.

In order to give a final application of the connectedness of S_o, another lemma is necessary. The following result is much more difficult than Lemma 5.9 and is proved in [86].

LEMMA 5.10. (See [86]). *Assume that* f *and* g *satisfy A1, that* $g(x) = x$ *and that* $1 = g'(0) < f'(0) = c$. *If* S *is as in eq. (5.57), there exists a constant* M *such that* $z_2(\lambda,\varphi) \leq 2 + M\lambda^{-1}$ *for all* $(\lambda,\varphi) \in S$.

It is likely that a version of Lemma 5.10 is true for general g, but I shall not discuss this point.

Lemma 5.9 implies that the map $(\lambda,\varphi) \in S \to z_2(\lambda,\varphi)$ is continuous and $z_2(\lambda_o, 0) = \frac{2\pi}{\nu_o}$ and Lemma 5.10 yields

$$\lim_{\substack{\lambda\to\infty \\ (\lambda,\varphi)\in S}} z_2(\lambda,\varphi) = 2 ,$$

so by connectivity of S_o it must be true that for every number p, $2 < p < \frac{2\pi}{\nu_o}$ there exists $\lambda > 0$ and a slowly oscillating periodic solution of eq. $(5.1)_\lambda$ of minimal period p.

PROPOSITION 5.1. (See [86]). *Let notation and assumptions be as in Theorem 5.1 and suppose in addition that* $g(x) = x$. *Then for every* p *such that* $2 < p < \frac{2\pi}{\nu_0}$, *there exists* $\lambda > 0$ *and a slowly oscillating periodic solution of minimal period* p .

REMARK 5.5. In applications it may be a nontrivial problem to determine whether the hypotheses of Theorem 5.1 are satisfied. Consider, for example, the equation

$$\varepsilon \dot{y}(t) = -y(t) + \mu(1-\sin(y(t-1))) \ , \ \varepsilon,\mu > 0 \ , \tag{5.60}$$

which arises in optics. If $y_\mu \in (0, \frac{\pi}{2})$ denotes the unique solution of

$$-y_\mu + \mu(1-\sin(y_\mu)) = 0 \ , \ 0 < y_\mu < (\frac{\pi}{2})$$

one is interested in periodic solutions of equation (5.60) which oscillate slowly about y_μ . If one defines $x(t)$ by $y(t) = y_\mu + x(t)$, one wants slowly oscillating periodic solutions of

$$\varepsilon \dot{x}(t) = -x(t) - f_\mu(x(t-1)) \ ,$$

where

$$f_\mu(x) = y_\mu - \mu(1-\sin(y_\mu + x)) \ .$$

The question becomes for what range of $\mu > 0$ is condition A1 satisfied by $g(x) = x$ and $f_\mu(x)$ and also $f'_\mu(0) > 1$? One can prove (see [86]) that (to four decimal places) the answer is

$$1.1773 < \mu < 2.3879 \ .$$

The precise answer is given in terms of solutions of some transcendental equations.

REFERENCES

[1] Abraham, R. and Robbin, J., *Transversal Mappings and Flows*, W.A. Benjamin, Inc., New York, 1967.

[2] Alexandroff, P. and Hopf, H., *Topologie*, Berlin, 1935.

[3] Alexander, J.C., A primer on connectivity, in: *Fixed Point Theory*, E. Fadell and G. Fournier, eds., Springer Lecture Notes in Mathematics, vol. 886, Springer-Verlag, New York, 1981, pp. 455-484.

[4] Amann, H., Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Review* 18(1976), 620-709.

[5] Amann, H., On the number of solutions of nonlinear equations in ordered Banach spaces, *J. Functional Analysis* 11(1972), 346-384.

[6] Amann, H., Fixed points of asymptotically linear maps in ordered Banach spaces, *J. Functional Analysis* 14(1973), 162-171.

[7] Amann, H., and Weiss, S., On the uniqueness of the topological degree, *Math. Z.* 130(1973), 39-54.

[8] Arens, R. and Eells, J., On embedding uniform and topological spaces, *Pacific J. Math.* 6(1956), 397-403.

[9] Baxter, N. and Nussbaum, R.D., A nonlinear integral equation, *Nonlinear Analysis* 5(1981), 1285-1307.

[10] Blythe, S.P., Nisbet, R.M. and Gurney, W.S.C., Instability and complex dynamic behaviour in population models with long time delays, *Theor. Pop. Biol.* 2(1982), 147-176.

[11] Bonsall, F.F., Linear operators in complete positive cones, *Proc. London Math. Soc.* 8(1958), 53-75.

[12] Bonsall, F.F., Positive operators compact in an auxiliary topology, *Pacific J. Math.* 10(1960), 1131-1138.

[13] Bonsall, F.F., *Lectures on Some Fixed Point Theorems of Functional Analysis*, Tata Institute of Fundamental Research, Bombay, 1962.

[14] Borsuk, K., *Theory of Retracts*, Polish Sci. Publ. Warsaw, 1967.

[15] Brown, R.F., An elementary proof of the uniqueness of the fixed point index, *Pacific J. Math.* 35(1970), 549-558.

[16] Brown, R.F., *The Lefschetz Fixed Point Theorem*, Scott Foresman Co., Glenview, Illinois, 1971.

[17] Browder, F.E., On the fixed point index for continuous mappings of locally connected spaces. *Summa Brasil Math.* 4(1960), 253-293.

[18] Browder, F.E., On continuity of fixed points under deformation of continuous mapping, *Summa. Brasil. Math.* 4(1960), 183-191.

[19] Browder, F.E. and Nussbaum, R.D., The topological degree for noncompact nonlinear mappings in Banach spaces, *Bull. Amer. Math. Soc.* 74(1968), 671-676.

[20] Browder, F.E., Local and global properties of nonlinear mappings in Banach spaces, *Instituto Naz. di Alta Mat. Symposia Math.* 2(1968), 13-35.

[21] Browder, F.F., On a generalization of the Schauder fixed point theorem, *Duke Math. J.* 26(1959), 291-303.

[22] Browder, F.E., A further generalization of the Schauder fixed point theorem, *Duke Math. J.* 32(1965), 575-578.

[23] Browder, F.E., Asymptotic fixed point theorems, *Math. Ann.* 185(1970), 38-60.

[24] Bumby, R.T., Hausdorff dimensions of Cantor sets, *J. Reine Angew. Math.* 331 (1982), 192-206.

[25] Chapin, S. and Nussbaum, R.D., Asymptotic estimates for the periods of periodic solutions of a differential-delay equation, *Michigan Math. J.* 31(1984), 215-229.

[26] Chow, S.-N. and Green, D. Jr., Some results on singular delay-differential equations, in: *Chaos in Dynamical Systems;* Proceedings of the Guelph Conferences, W.R. Smith and P. Fischer, eds., Marcel Dekker, to appear.

[27] Chow, S.-N. and Mallet-Paret, J., Singularly perturbed delay-differential equations, in: *Coupled Nonlinear Oscillators*, J. Chandra and A.C. Scott, eds., North-Holland Math. Studies, vol. 80(1983), pp. 7-12.

[28] Cooke, K. and Yorke, J., Equations modelling population growth and gonorrhea epidemiology, *Math. Biosci.* 16(1973), 75-101.

[29] Cooke, K.L. and Kaplan, J.L., A periodicity threshold theorem for epidemics and population growth, *Math. Biosci.* 31(1976), 87-105.

[30] Cronin, J., *Fixed Points and Topological Degree in Nonlinear Analysis*, American Math. Soc., Providence, Rhode Island, 1964.

[31] Dancer, E.N., Global solution branches for positive mappings, *Arch. Rat. Mech. Anal.* 52(1973), 181-192.

[32] Dancer, E.N., Solution branches for mappings in cones, and applications, *Bull. Aust. Math. Soc.* 11(1974), 131-143.

[33] Darbo, G., Punti uniti in trasformazioni a condiminio non compatto, *Rend. Sem. Math. Univ. Padova* 24(1955), 84-92.

[34] DeFigueiredo, D., Lions, P.-L. and Nussbaum, R.D., Estimations à priori pour les solutions positives de problèmes elliptiques semi linéaires, *C.R. Acad. Sci. Paris*, série A, 1980.

[35] DeFigueiredo, D., Lions, P.-L. and Nussbaum, R.D., A priori estimates and existence of positive solutions of semilinear elliptic equations, *J. Math. Pures Appl.* 61(1982), 41-63.

[36] Deleanu, A., Théorie des points fixes sur les retractes de voisinage des espaces convexoïdes, *Bull. Soc. Math. France* 89(1959), 235-243.

[37] Derstine, M.W., Gibbs, H.M., Hopf, F.A. and Kaplan, D.L., Alternative paths to chaos in optical bistability, *Phys. Rev.* A 27(1983), 3200-3208.

[38] Dold, A., Fixed point index and fixed point theorems for euclidean neighborhood retracts, *Topology* 4(1965), 1-8.

[39] Dugundji, J., An extension of Tietze's theorem, *Pacific J. Math.* 1(1951), 353-367.

[40] Dugundji, J. and Granas, A., *Fixed Point Theory*, vol. 1, Polska Adademia Nauk, Monografie Matematyczne, Tom 61, Warsaw, 1982.

[41] Edmunds, D.A., Potter, A.J.B. and Stuart, C.A., Noncompact positive Operators, *Proc. Royal Soc. London* A 328(1972), 67-81.

[42] Eells, J. and Fournier, G., La théorie des points fixes des applications à itérée condensante, *Bull. Soc. Math. France*, Mémoire 45(1976), 91-120.

[43] Fenske, C., Analytische Theorie des Abbildungsgrades für Abbildungen in Banachraümen, *Math. Nachr.* 48(1971), 279-290.

[44] Fenske, C. and Peitgen, H.-O., Fixed points of zero index in asymptotic fixed point theory, *Pacific J. Math.* 66(1976), 391-410.

[45] Fournier, G. and Peitgen, H.-O., On some fixed point principles for cones in linear normed spaces, *Math. Ann.* 225(1977), 205-218.

[46] Fournier, G. and Peitgen, H.-O., Leray endomorphisms and cone maps. *Ann. Scuola Norm. Sup. Pisa* IV, vol. 5(1978), 149-179.

[47] Fournier, G., Généralisations du théorème de Lefschetz pour des espaces non compacts. II., *Bull. Acad. Pol. Sci.* 23(1975), 701-706.

[48] Führer, L., Ein elementarer analytischer Beweis zur Eindeutigkeit des Abbildungsgrades im $\mathbf{R}^n$, *Math. Nachr.* 54(1972), 259-267.

[49] Gibbs, H.M., Hopf, F.A., Kaplan, D.L. and Shoemaker, R.L., Observation of chaos in optical bistability, *Phys. Rev. Lett.* 46(1981), 474-477.

[50] Gohberg, I.T., Goldenstein, L.S. and Markus, A.S., Investigation of some properties of bounded linear operators in connection with their q-norms (in Russian), *Uch. Zap. Kishinevsk. Un-ta* 29(1957), 29-36.

[51] Goldenstein, L.S. and Markus, A.S., On a measure of noncompactness of bounded sets and linear operators, *Studies in Algebra and Math. Anal.* (Russian), pp. 45-54, Izdat. "Karta Moldovenjaske", Kishinev, 1965 (Math. Reviews, vol. 35(1968), Rev. no. 789).

[52] Grafton, R.B., A periodicity theorem for autonomous functional differential equations, *J. Differential Equations* 6(1969), 87-109.

[53] Grafton, R.B., Periodic solutions of certain Liénard equations with delay, *J. Differential Equations* 11(1972), 519-527.

[54] Granas, A., Theory of compact vector fields, *Rozprawy Mat.* 30(1962), 93 pp.

[55] Granas, A., Some theorems in fixed point theory, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* 17(1969), 131-137.

[56] Granas, A., The Leray-Schauder index and the fixed point theory for arbitrary ANR's, *Bull. Soc. Math. France* 100(1972), 209-228.

[57] Gurney, W.S.C., Blythe, S.P. and Nisbet, R.M., Nicholson's blowflies revisited, *Nature* 287(1980), 17-21.

[58] Hadeler, K.P., Tomiuk, J., Periodic solutions of difference-differential equations, *Arch. Rat. Mech. Anal.* 65(1977), 87-95.

[59] Hale, J., *Theory of Functional Differential Equations*, Springer Verlag, New York, Heidelberg, Berlin, 1977.

[60] Hanner, O., Some theorems on absolute neighborhood retracts, *Arkiv for Matematik* 1(1951), 389-408.

[61] Hanner, O., Retraction and extension of mappings of metric and non-metric spaces, *Arkiv for Matematik* 2(1952-1954), 315-360.

[62] an der Heiden, U. and Mackey, M.C., The dynamics of production and destruction: analytic insight into complex behaviour, *J. Math. Biol.* 16(1982), 75-101.

[63] Hopf, F., Kaplan, D., Gibbs, H. and Shoemaker, P., Bifurcation to chaos in optical bistability, *Phys. Rev.* A 25(1982), 2172-2182.

[64] Hu, S.T., *Theory of Retracts*, Wayne State University Press, Detroit, 1965.

[65] Ikeda, K., Multiple-valued stationary state and its instability of the transmitted light by a ring cavity system, *Opt. Comm.* 30(1979), 257-261.

[66] Ikeda, K., Daido, H. and Akimoto, O., Optical turbulence: chaotic behaviour of transmitted light from a ring cavity, *Phys, Rev. Lett.* 45(1980), 709-712

[67] Ikeda, K., Kondo, K. and Akimoto, O., Successive higher-harmonic bifurcations in systems with delayed feedback, *Phys. Rev. Lett.* 49(1982), 1467-1470.

[68] Jones, G.S., The existence of periodic solutions of $f'(x) = -\alpha f(x-1)[1+f(x)]$, *J. Math. Anal. Appl.* 4(1962), 440-469.

[69] Jones, G.S., Periodic motions in Banach space and applications to functional differential equations, *Contributions to Differential Equations* 3(1964), 75-106.

[70] Kaplan, J. and Yorke, J., On the stability of a periodic solution of a differential delay equations, *SIAM J. Math. Anal.* 6(1975), 268-282.

[71] Kaplan, J. and Yorke, J., On the nonlinear differential delay equation $x'(t) = -f(x(t),x(t-1))$, *J. Differential Equations* 23(1977), 293-314.

[72] Krasnosel'skii, M.A., Fixed points of cone-compressing or cone-extending operators, *Soviet. Math. Dokl.* 1(1960), 1285-1288.

[73] Krasnosel'skii, M.A., *Topological Methods in the Theory of Nonlinear Integral Equations*, MacMillan Co., New York, 1964.

[74] Krasnosel'skii, M., *Positive Solutions of Operator Equations*, P. Noordhoff Ltd., Groningen, 1964.

[75] Krasnosel'skii, M. and Zabreiko, P.P., Iterations of operators and the fixed point index, *Soviet Math. Doklady* 12(1971), 294-298.

[76] Krein, M.G. and Rutman, M.A., Linear operators leaving invariant a cone in a Banach space, *Amer. Math. Soc. Transl.* no. 26(ser. 1).

[77] Kuratowski, C., Sur les espaces complets, *Fund. Math.*, 15(1930), 301-309.

[78] Kuratowski, C., *Topology*, vol. 2, Academic Press, New York, 1968.

[79] Lasota, A. and Wazewska-Czyzewska, M., Mathematical models of the red cell system (Polish), *Mate. Stosowana* 6(1976), 25-40.

[80] Leray, J. and Schauder, J., Topologie et équations fonctionnelles, *Ann. Sci. Ecole Norm. Sup.* 51(1934), 45-78.

[81] Leray, J., Sur les équations et les transformations, *J. Math. Pures Appl.* 24(1945), 95-167.

[82] Leray, J., Théorie des points fixes: indice total et nombre de Lefschetz, *Bull. Soc. Math. France* 87(1959), 221-233.

[83] Mackey, M.C., Periodic auto-immune hemolytic anemia: an induce dynamical desease, *Bull. Math. Biol.* 41(1979), 829-234.

[84] Mackey, M.C. and an der Heiden, U., Dynamical diseases and bifurcations: understanding functional disorders in physiological systems, *Funkt. Biol. Med.* 156(1982), 156-164.

[85] Mackey, M. and Glass, L., Oscillation and chaos in physiological control systems, *Science* 197(1977), 287-289.

[86] Mallet-Paret, J. and Nussbaum, R.D., Global continuation and asymptotic behaviour for periodic solutions of a differential-delay equation, submitted for publication.

[87] Mallet-Paret, J. and Nussbaum, R.D., Global continuation and complicated trajectories for periodic solutions of a differential-delay equation, to appear in the Proceedings of the 1983 A.M.S. Summer Symposium on Nonlinear Functional Analysis (Berkeley, California).

[88] Nagumo, M., A theory of degree based on infinitesimal analysis, *Amer. J. Math.* 73(1951), 485-496.

[89] Nirenberg, L., *Topics in Nonlinear Functional Analysis*, N.Y.U. Courant Institute Lecture Notes, New York, 1974.

[90] Nussbaum, R.D., The fixed point index for local condensing maps, *Ann. Mat. Pura Appl.* 89(1971), 217-258.

[91] Nussbaum, R.D., Some asymptotic fixed point theorems, *Transactions A.M.S.* 171(1972), 349-375.

[92] Nussbaum, R.D., The fixed point index and asymptotic fixed point theorems for k-set-contractions, *Bull. A.M.S.* 75(1969), 490-495.

[93] Nussbaum, R.D., Asymptotic fixed point theorems for local condensing maps, *Math. Ann.* 191(1971), 181-195.

[94] Nussbaum, R.D., Degree theory for local condensing maps, *J. Math. Anal. Appl.* 37(1972), 741-766.

[95] Nussbaum, R.D., Generalizing the fixed point index, *Math. Ann.* 228(1977), 259-278.

[96] Nussbaum, R.D., Periodic solutions of some nonlinear autonomous functional differential equations, *Ann. Mat. Pura Appl.* 101(1974), 263-306.

[97] Nussbaum, R.D., Periodic solutions of some nonlinear autonomous functional differential equations, II., *J. Differential Equations* 14(1973), 360-394.

[98] Nussbaum, R.D., A correction of "Periodic solutions of some nonlinear autonomous functional differential equations II", *J. Differential Equations* 16(1974), 548-549.

[99] Nussbaum, R.D., A global bifurcation theorem with applications to functional differential equations, *J. Functional Analysis* 19(1975), 319-338.

[100] Nussbaum, R.D., The range of periods of periodic solutions of $x'(t) = -\alpha f(x(t-1))$, *J. Math. Anal. Appl.* 58(1977), 280-292.

[101] Nussbaum, R.D., Uniqueness and nonuniqueness for periodic solutions of $x'(t) = -g(x(t-1))$, *J. Differential Equations* 34(1979), 24-54.

[102] Nussbaum, R.D., Periodic solutions of nonlinear, Autonomous functional differential equations, in: *Functional Differential Equations and Approximation of Fixed Points*, H.-O. Peitgen and H.-O. Walther, eds., Springer Verlag Lecture Notes in Mathematics, vol. 730, Springer Verlag, New York, 1979, pp. 283-325.

[103] Nussbaum, R.D., Integral Equations from the theory of epidemics, in: *Nonlinear Systems and Applications*, B. Lakshmikantham, ed., Academic Press, New York, 1977, pp. 235-255.

[104] Nussbaum, R.D., Periodic solutions of some nonlinear integral equations, in: *Dynamical Systems*, A. Bednarek and L. Cesari, eds., Academic Press, New York, 1977, pp. 221-251.

[105] Nussbaum, R.D., A periodicity threshold theorem for some nonlinear integral equations, *SIAM J. Math. Anal.* 9(1978), 356-376.

[106] Nussbaum, R.D., The radius of the essential spectrum, *Duke Math. J.* 38(1970), 473-478.

[107] Nussbaum, R.D., Eigenvectors of nonlinear positive operators and the linear Krein-Rutman theorem, in: *Fixed Point Theory*, E. Fadell and G. Fournier, eds., Springer Lecture Notes in Mathematics, vol. 886, Springer Verlag, New York 1981. pp. 309-331.

[108] Peitgen, H.-O., On the Lefschetz number for iterates of continuous mappings, *Proc. A.M.S.* 54(1976), 441-444.

[109] Rabinowitz, P.H., Some global results for non-linear eigenvalue problems, *J. Functional Analysis* 7(1971), 487-513.

[110] Rabinowitz, P.H., Some aspects of nonlinear eigenvalue problems, *Rocky Mountain Math. J.* 3(1973), 161-202.

[111] Rabinowitz, P.H., Théorie du degré topologique et applications à des problèmes aux limites non linéaires, Notes by H. Berestycki, from a course at Université Paris VI, Spring 1973.

[112] Sadovskii, B.N., On a fixed point principle, *Functional Analysis and Appl.* 1(1967), 74-76.

[113] Sadovskii, B.N., Limit compact and condensing operators, *Russian Math. Surveys* 27(1972), 85-155.

[114] Schaefer, H.H., On nonlinear positive operators, *Pacific J. Math.* 9(1959), 847-860.

[115] Schaefer, H.H., Some spectral properties of positive linear operators, *Pacific J. Math.* 10(1960), 1009-1019.

[116] Schaefer, H.H., *Topological Vector Spaces*, Springer Verlag, New York, 1971.

[117] Schwartz, J.T., *Nonlinear Functional Analysis*, Gordon & Breach, New York, 1969.

[118] Steinlein, H., Ein Satz über den Leray-Schauderschen Abbildungsgrad, *Math. Z.* 120(1972), 176-208.

[119] Steinlein, H., Über die verallgemeinerten Fixpunktindizes von Iterierten verdichtender Abbildungen, *Manuscripta Math.* 8(1972), 251-266.

[120] Stuart, C.A., Some bifurcation theory for k-set-contractions, *Proc. London Math. Soc.* 27(1973), 531-550.

[121] Thompson, R.B., A unified approach to local and global fixed point indices, *Advances in Math.* 3(1969), 1-72.

[122] Tromba, A.J., Fredholm vector fields and a transversality theorem, *J. Functional Analysis* 23(1976), 362-368.

[123] Tromba, A.J., The beer barrel theorem, in: *Functional Differential Equations and Approximation of Fixed Points*, H.-O. Peitgen and H.-O. Walther, eds., Springer Verlag Lecture Notes in Mathematics, vol. 730, pp. 484-488.

[124] Turner, R.E.L., Transversality and cone maps, *Arch. Rational Mech. Anal.* 58(1975), 151-179.

[125] Whyburn, G.T., *Topological Analysis*, Princeton Univ. Press, Princeton, N.J., 1958.

[126] Wright, E.M., A nonlinear difference-differential equation, *J. Reine Angew. Math.* 194(1955), 66-87.

LES PRESSES DE L'UNIVERSITÉ DE MONTRÉAL
C.P. 6128, succ. «A», Qué., Canada H3C 3J7

EXTRAIT DU CATALOGUE

Mathématiques

COLLECTION «SÉMINAIRE DE MATHÉMATIQUES SUPÉRIEURES»

1. **Problèmes aux limites dans les équations aux dérivées partielles.** Jacques L. LIONS
2. **Théorie des algèbres de Banach et des algèbres localement convexes.** Lucien WAELBROECK
3. **Introduction à l'algèbre homologique.** Jean-Marie MARANDA
4. **Séries de Fourier aléatoires.** Jean-Pierre KAHANE
5. **Quelques aspects de la théorie des entiers algébriques.** Charles PISOT
6. **Théorie des modèles en logique mathématique.** Aubert DAIGNEAULT
7. **Promenades aléatoires et mouvements brownien.** Anatole JOFFE
8. **Fondements de la géométrie algébrique moderne.** Jean DIEUDONNÉ
9. **Théorie des valuations.** Paulo RIBENBOIM
10. **Catégories non abéliennes.** Peter HILTON, Tudor GANEA, Heinrich KLEISLI, Jean-Marie MARANDA, Howard OSBORN
11. **Homotopie et cohomologie.** Beno ECKMANN
12. **Intégration dans les groupes topologiques.** Geoffrey FOX
13. **Unicité et convexité dans les problèmes différentiels.** Shmuel AGMON
14. **Axiomatique des fonctions harmoniques.** Marcel BRELOT
15. **Problèmes non linéaires.** Félix E. BROWDER
16. **Équations elliptiques du second ordre à coefficients discontinus.** Guido STAMPACCHIA
17. **Problèmes aux limites non homogènes.** José BARROS-NETO
18. **Équations différentielles abstraites.** Samuel ZAIDMAN
19. **Équations aux dérivées partielles.** Robert CARROL, George F.D. DUFF, Jöran FRIBERG, Jules GOBERT, Pierre GRISVARD, Jindrich NECAS et Robert SEELEY
20. **L'Algèbre logique et ses rapports avec la théorie des relations.** Roland FRAÏSSÉ
21. **Logical Systems Containing Only a Finite Number of Symbols.** Leon HENKIN
24. **Représentabilité et définissabilité dans les algèbres transformationnelles et dans les algèbres polyadiques.** Léon LEBLANC
25. **Modèles transitifs de la théorie des ensembles de Zermelo-Fraenkel.** Andrzej MOSTOWSKI
26. **Théorie de l'approximation des fonctions d'une variable complexe.** Wolfgang H.J. FUCHS
27. **Les Fonctions multivalentes.** Walter K. HAYMAN
28. **Fonctionnelles analytiques et fonctions entières (n variables).** Pierre LELONG
29. **Applications of Functional Analysis to Extremal Problems for Polynomials.** Qazi Ibadur RAHMAN
30. **Topics in Complex Manifolds.** Hugo ROSSI
31. **Théorie de l'inférence statistique robuste.** Peter J. HUBER
32. **Aspects probabilistes de la théorie du potentiel.** Mark KAC
33. **Théorie asymptotique de la décision statistique.** Lucien M. LECAM
34. **Processus aléatoires gaussiens.** Jacques NEVEU
35. **Nonparametric Estimation.** Constance van EEDEN
36. **K-Théorie.** Max KAROUBI
37. **Differential Complexes.** Joseph J. KOHN
38. **Variétés hilbertiennes : aspects géométriques.** Nicolaas H. KUIPER
39. **Deformations of Compact Complex Manifold.** Masatake KURANISHI
40. **Grauert's Theorem of Direct Images of Coherent Sheaves.** Raghavan NARASIMHAN
41. **Systems of Linear Partial Differential Equations and Deformation of Pseudogroup Structures.** A. KUMPERA et D.C. SPENCER
42. **Analyse globale.** P. LIBERMANN, K.D. ELWORTHY, N. MOULIS, K.K. MUKHERJEA, N. PRAKASH, G. LUSZTIC et W. SHIH
43. **Algebraic Space Curves.** Sheeram S. ABHYANKAR
44. **Théorèmes de représentabilité pour les espaces algébriques.** Michael ARTIN
45. **Groupes de Barsotti-Tate et cristaux de Dieudonné.** Alexandre GROTHENDIECK
46. **On Flat Extensions of a Ring.** Massayoshi NAGATA
47. **Introduction à la théorie des sites et son application à la construction des préschémas quotients.** Masayoshi MIYANISHI
48. **Méthodes logiques en géométrie diophantienne.** Shuichi TAKAHASHI
49. **Index Theorems of Atiyah — Bott — Patodi and Curvature Invariants.** Ravindra S. KULKARNI
50. **Numerical Methods in Statistical Hydrodynamics.** Alexandre CHORIN
51. **Introduction à la théorie des hypergraphes.** Claude BERGE
52. **Automath, a Language for Mathematics.** Nicolaas G. DE BRUIJN
53. **Logique des topos (Introduction à la théorie des topos élémentaires).** Dana SCHLOMIUK
54. **La Série génératrice exponentielle dans les problèmes d'énumération.** Dominique FOATA
55. **Feuilletages : résultats anciens et nouveaux (Painlevé, Hector et Martinet).** Georges H. REEB

56. **Finite Embedding Theorems for Partial Designs and Algebras.** Charles C. LINDNER et Trevor EVANS
57. **Minimal Varieties in Real and Complex Geometry.** H. Blaine LAWSON, Jr
58. **La Théorie des points fixes et ses applications à l'analyse.** Kazimierz GEBA, Karol BORSUK, Andrzej JANKOWSKI et Edward ZHENDER
59. **Numerical Analysis of the Finite Element Method,** Philippe G. CIARLET
60. **Méthodes numériques en mathématiques appliquées.** J.F. Giles AUCHMUTY, Michel CROUZEIX, Pierre JAMET, Colette LEBAUD, Pierre LESAINT et Bertrand MERCIER
61. **Analyse numérique matricielle.** Paul ARMINJON
62. **Problèmes d'optimisation en calcul des probabilités.** Serge DUBUC
63. **Chaînes de Markov sur les permutations.** Gérard LETAC
64. **Géométrie différentielle stochastique.** Paul MALLIAVIN
65. **Numerical Methods for Solving Time-Dependent Problems for Partial Differential Equations.** Heinz-Otto KREISS
66. **Difference Sets in Elementary Abelian Groups.** Paul CAMION
67. **Groups in Physics : Collective Model of the Nucleus; Canonical Transformation in Quantum Mechanics.** Marcos MOSHINSKY
68. **Points fixes pour les applications compactes : espaces de Lefschetz et la théorie de l'indice.** Andrzej GRANAS
69. **Set Theoretic Methods in Homological Algebra and Abelian Groups.** Paul EKLOF
70. **Abelian p-Groups and Mixed Groups.** Laszlo FUCHS
71. **Integral Representations and Structure of Finite Group Rings.** Klaus W. ROGGENKAMP
72. **Homological Invariants of Modules over Commutative Rings.** Paul ROBERTS
73. **Representations of Valued Graphs.** Vlastimil DLAB
74. **Groupes abéliens sans torsion.** Khalid BENABDALLAH
75. **Lie Groups, Lie Algebras and Representation Theory.** Hans ZASSENHAUS
76. **Birational Geometry for Open Varieties.** Shigeru IITAKA
77. **Lectures on Hilbert Modular Surfaces.** Friedrich HIRZEBRUCH, Gerard van der GEER
78. **Complex Geometry in Mathematical Physics.** R.O. WELLS, Jr.
79. **Lectures on Approximation and Value Distribution.** Tord GANELIUS, Walter K. KAYMAN, Donald J. NEWMAN
80. **Sur la topologie des surfaces complexes compactes.** Srinivasacharyulu KILAMBI, Gottfried BARTHEL, Ludger KAUP
81. **Topics in Polynomial and Rational Interpolation and Approximation.** Richard S. VARGA
82. **Approximation uniforme qualitative sur des ensembles non bornés.** Paul M. GAUTHIER, Walter HENGARTNER
83. **Convolutions in Geometric Function Theory.** Stephan RUSCHEWEYH.
84. **Characteristic Properties of Quasidisks.** Frederick W. GEHRING.
85. **Curves in Projective Space.** Joe HARRIS.
86. **Les Inégalités de Markoff et de Bernstein.** Qazi Ibadur RAHMAN, Gerhard SCHMEISSER
87. **Analyse des données.** Francis CAILLIEZ
88. **Inférence statistique et analyse des données sous des plans d'échantillonnages complexes.** Carl E. SÅRNDAL
89. **Analyse de données chronologiques.** Guy MÉLARD
90. **Schauder's Estimates and Boundary Value Problems for Quasilinear Partial Differential Equations.** Manfred KÖNIG
91. **Topological Methods in Bifurcation Theory.** Kazimierz GĘBA et Paul H. RABINOWITZ
92. **Points fixes, points critiques et problèmes aux limites.** Jean MAWHIN
93. **Problèmes aux limites non linéaires pour certaines classes d'équations différentielles ordinaires.** Ronald B. GUENTHER

COLLECTION « CHAIRE AISENSTADT »

Physical Aspects of Lie Group Theory. Robert HERMANN
Quelques problèmes mathématiques en physique statistique. Mark KAC
La Transformation de Weyl et la fonction de Wigner : une forme alternative de la mécanique quantique. Sybren DE GROOT
Sur quelques questions d'analyse, de mécanique et de contrôle optimal. Jacques Louis LIONS
Mariages stables et leurs relations avec d'autres problèmes combinatoires. Donald E. KNUTH
Symétries, jauges et variétés de groupe. Yuval NE'EMAN
La Théorie des sous-gradients et ses applications à l'optimisation. Fonctions convexes et non convexes. R. Tyrrel ROCKAFELLAR

Achevé d'imprimer
en mai 1985 sur les presses
des Ateliers Graphiques Marc Veilleux Inc.
Cap-Saint-Ignace, Qué.